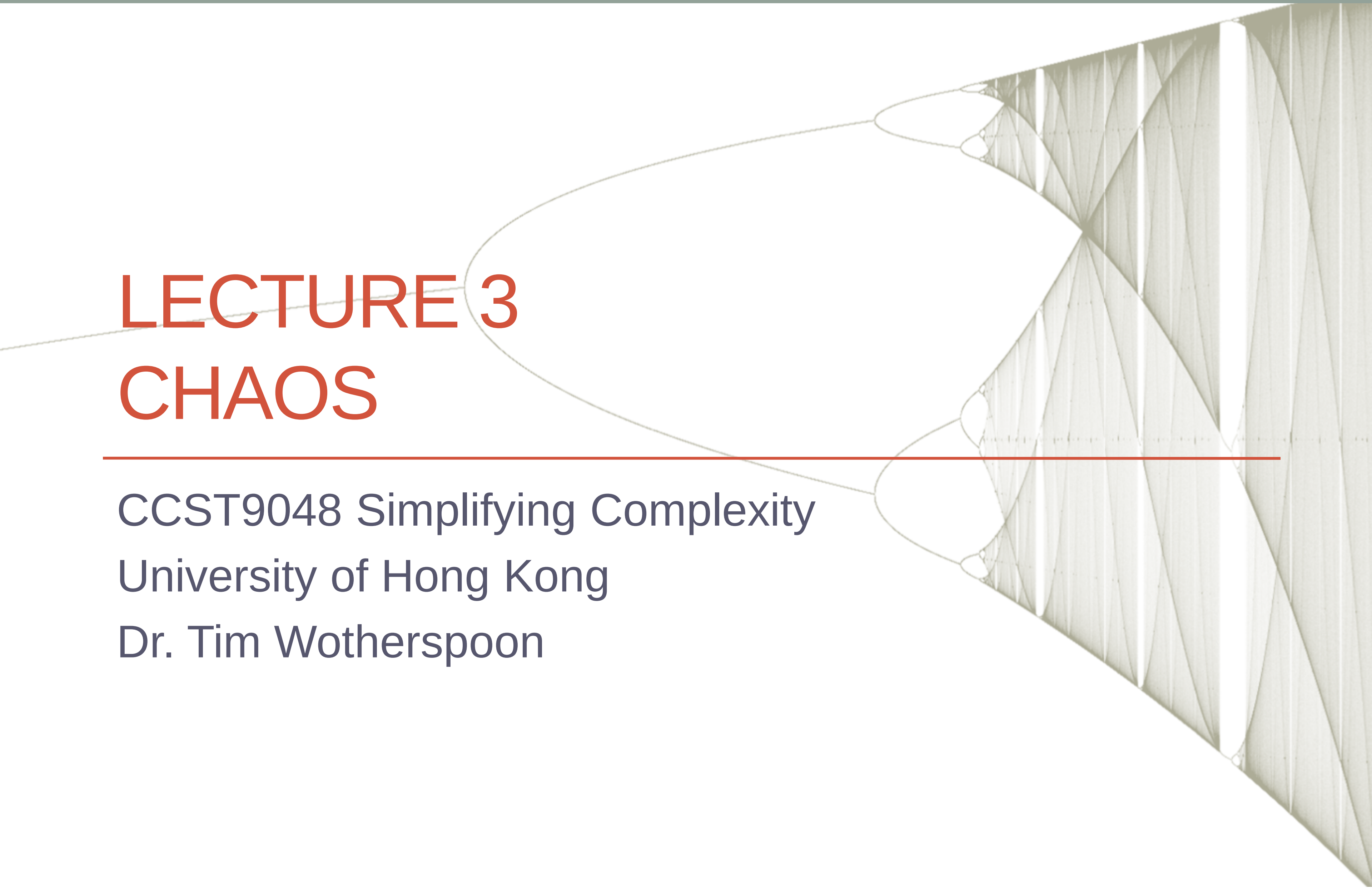
LECTURE 3
CHAOS

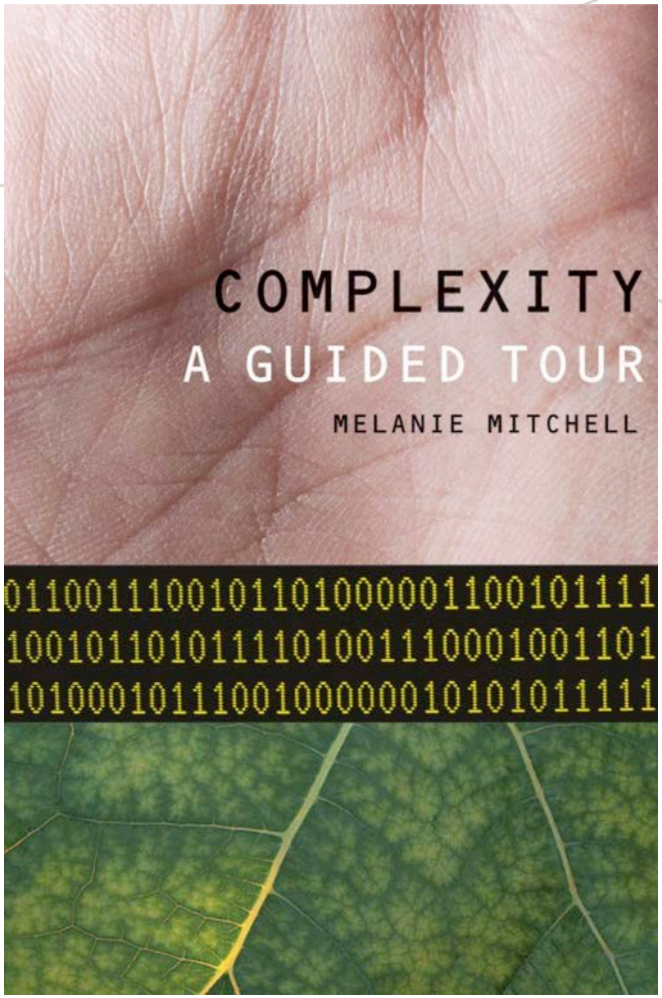


CCST9048 Simplifying Complexity

University of Hong Kong

Dr. Tim Wotherspoon

Complexity: A Guided Tour



- This lecture follows Chapter 2 very closely.
- You may find more information or a different explanation there if you are confused today.

Reductionism

“To divide all the difficulties under examination into as many parts as possible, and as many as were required to solve them in the best way.”

“to conduct my thoughts in a given order, beginning with the *simplest* and most easily understood objects, and gradually ascending, as it were step by step, to the knowledge of the most *complex.*”

From the *Discourse on the Method*, highlighted by Mitchell 2011

Definition

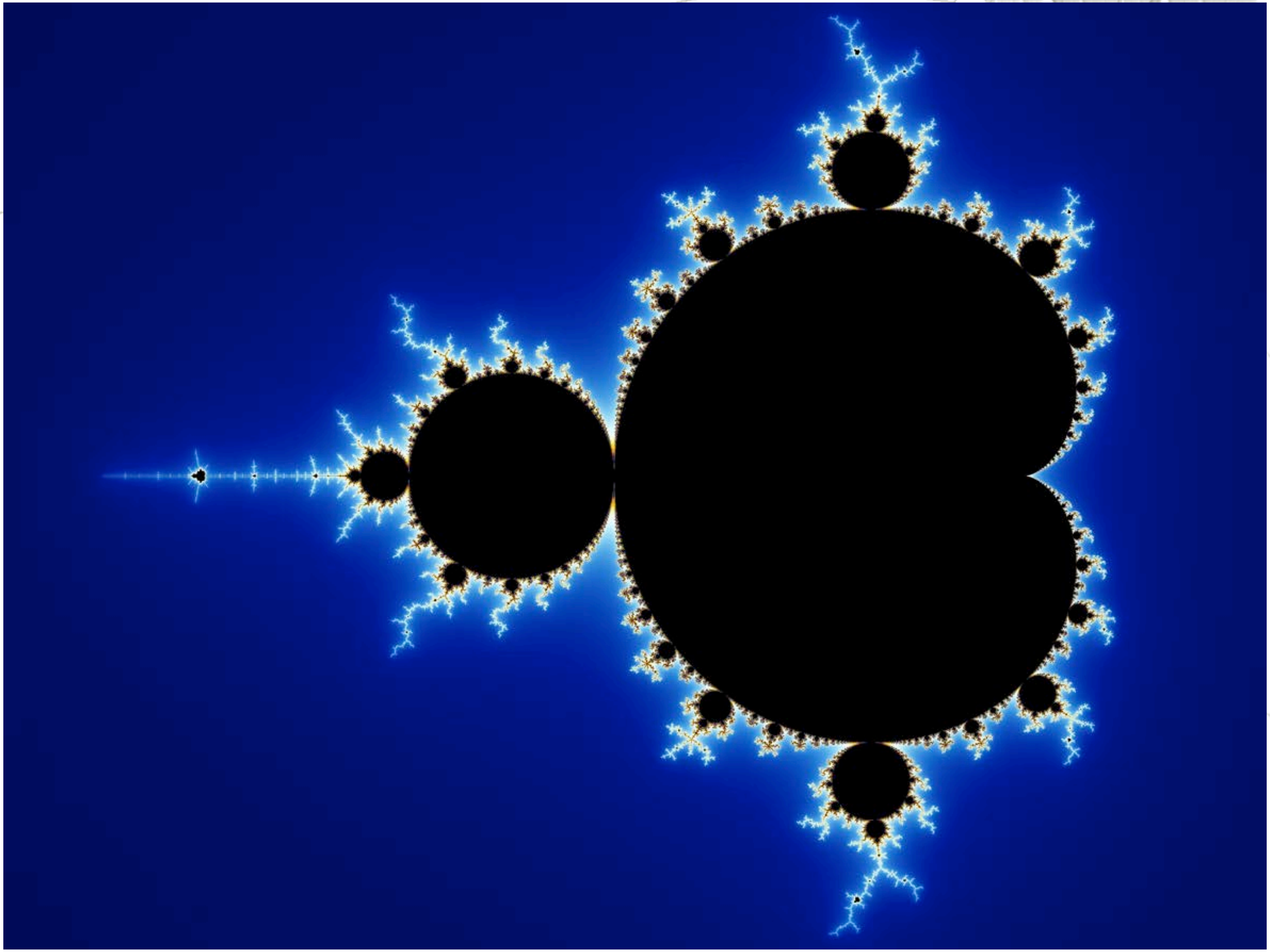
"Complex system: a system in which large networks of components with no central control and simple rules of operation give rise to complex collective behavior, sophisticated information processing and adaptation via learning or evolution."

Mitchell 2011

This is the Geometry Used in Nature!

http://photography.nationalgeographic.com/wallpaper/photography/photos/best-pod-april-2012/baja-california-rivers/

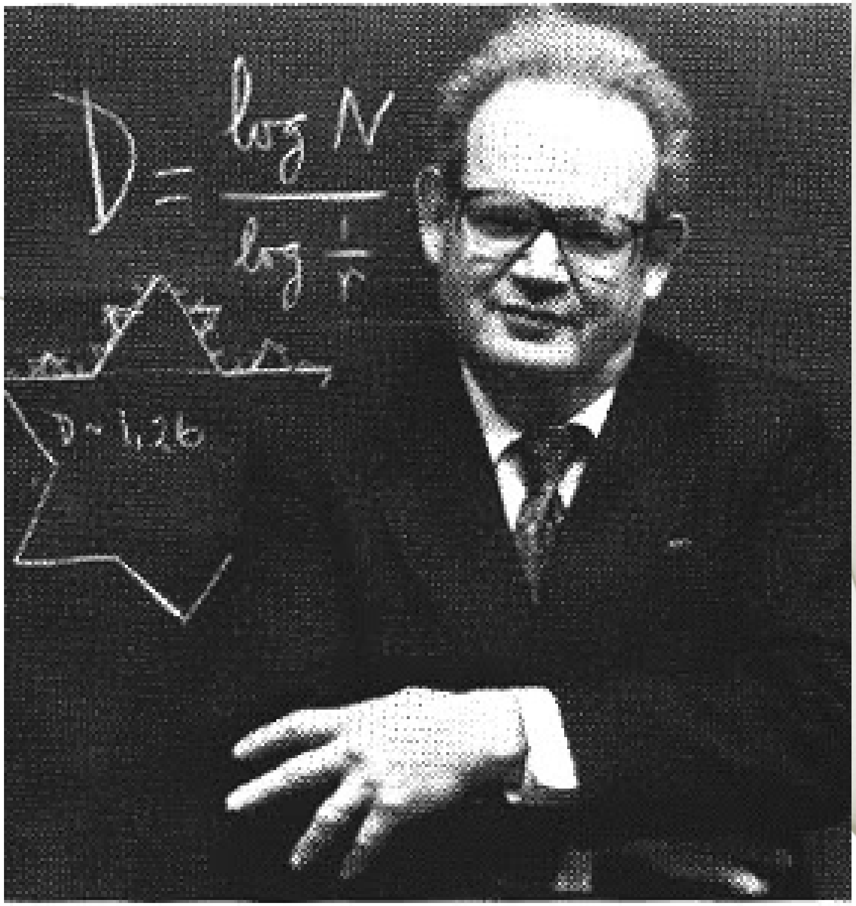
Mandelbrot Set



Julia and Mandelbrot Sets

"Bottomless wonders spring from simple rules which are repeated without end"

TED2010 [Fractals and the art of roughness]



Outline

- Brief History of Dynamics
- Graphical Analysis (Cobweb Diagram)
- Examples of Dynamical Systems
- Logistic Map
- Bifurcation Diagram
- Defining Chaos
- Lessons from Chaos

Brief History of Dynamics

- Dynamical systems refer to systems which show changes in time in some way.
- This includes the theory of mechanics dating back to Aristotle in Western Tradition.
- This work continues on through the work of Galileo and Newton.
- It is in no way limited to mechanics or even physics.

Incomplete List of Broad Names of Fields Concerned with Dynamics

- Physics
- Chemistry
- Biology
- Ecology
- Geology
- Environmental Science
- Economics
- Medicine
- Architecture
- Engineering
- Philosophy
- Psychology
- Sociology

- Any field that concerns itself with *time* in any fashion concerns itself with *Dynamics!*

Brief History of Dynamics

- Dynamical systems refer to systems which show changes in time in some way.
- A *Dynamical System* is a mathematical model that provides a theory for the basis of prediction of how some real system will *evolve* over time.
- Newton's Laws will make one such system.

$$\sum \vec{F}_{ext} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}$$

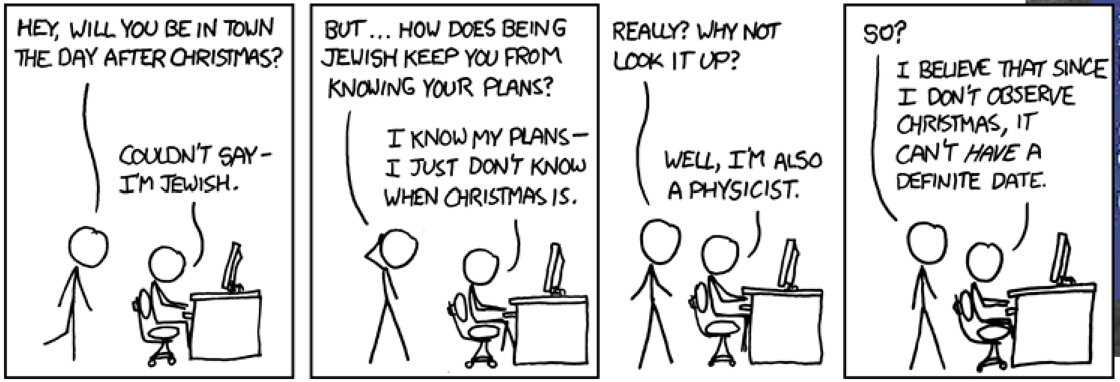
Brief History of Dynamics

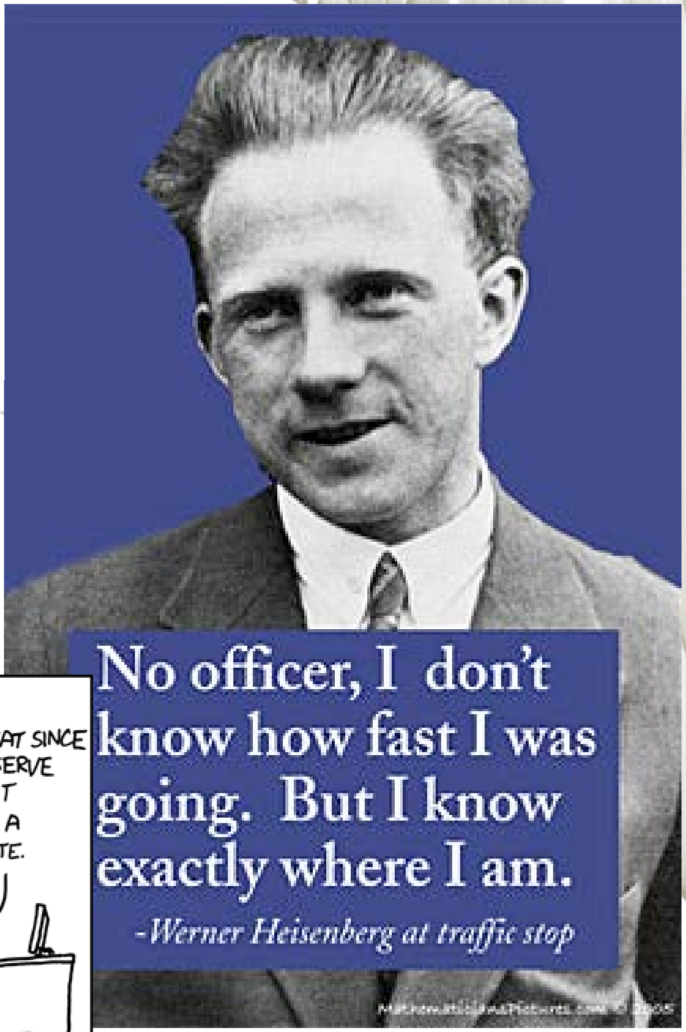
- Up until the end of the 19th century, many scientists believed that the world was *deterministic* and *predictable*.
- Systems of differential equations could be used in a *reductionist* manner to predict outcomes to *arbitrary precision*.

http://philosophicatz.wordpress.com/

Brief History of Dynamics--1927

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$





Brief History of Dynamics--1927

- "Heisenberg's Uncertainty Principle" shows that it is impossible to simultaneously know both the precise precision and velocity of a particle (described by the Schrödinger Equation).

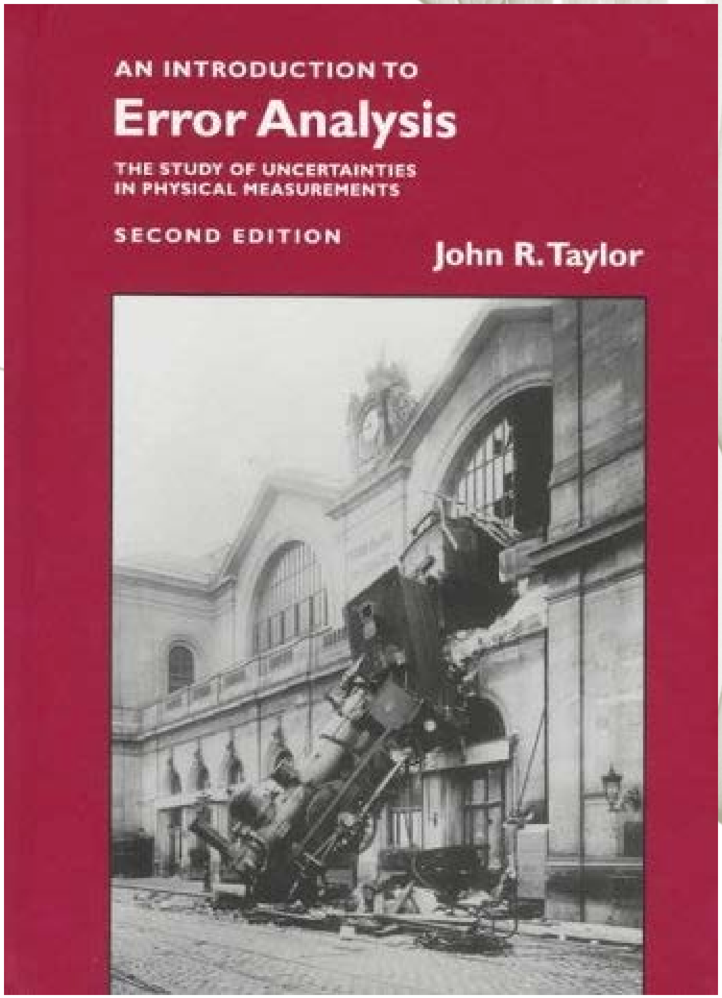
- However, it cannot matter for most of us!

$$\Delta x \Delta p \geq 5.27285863 \times 10^{-35} m^2 kg/s$$

- The engineers still have a way to go!
- Modern Scanning Transmission Electron Microscopes offer spatial resolution of only 1 x 10^{-10} m

Brief History of Dynamics

- Quantum Mechanics and Heisenberg's Uncertainty Principle tell us that fundamentally the universe is *probabilistic* and *unpredictable.*
- In many situations, for all practical purpose, the situation is still *deterministic.*
- We hope that *Error Analysis* will work. In this sense, a *deterministic* universe is still a *predictable* one.



Brief History of Dynamics

- If we drop a object from a height. We don't know exactly from what height we drop it.

$$h = 1.0m \pm 0.05m$$

- Mechanics tells us that

$$h = \frac{1}{2} g t^2$$

- So given that, we can predict the time to fall is between 0.44s and 0.46s.
- Of course, this theory is wrong. This equation goes back Galileo and was improved upon by Newton and then Einstein and we ignored air friction and several other effects.

Brief History of Dynamics

- In that example, the relationship between the uncertainty in the initial measurement and the uncertainty in the outcome is manageable.
- Much of our lives are this way.
- Are all things this way?

"influences whose physical magnitude is too small to be taken account of by a finite being, …may produce results of the highest importance."

James Clerk Maxwell--1873

Brief History of Dynamics

- A Hubble Telescope on an Earth with no atmosphere could measure the position of the moon in the sky to a precision of 0.1 arc seconds.
- If we knew that its *exact* distance from us, say 380,000km (in fact we know it's distance to a precision of a few mm due to the *Lunar Laser Ranging Experiment*). A Hubble Telescope could only tell us where was it was to about 200m of precision.
- Can we use Error Analysis to still make predictions about the moon?

Brief History of Dynamics

- 1887 - Attempted to expand Newton's work from explaining the motion of two celestial bodies to three.
- He failed spectacularly!
- In fact, he showed that starting from Newtonian Mechanics there was no way to algebraically solve the problem.
- He also discovered something about the problem was very interesting.

Brief History of Dynamics

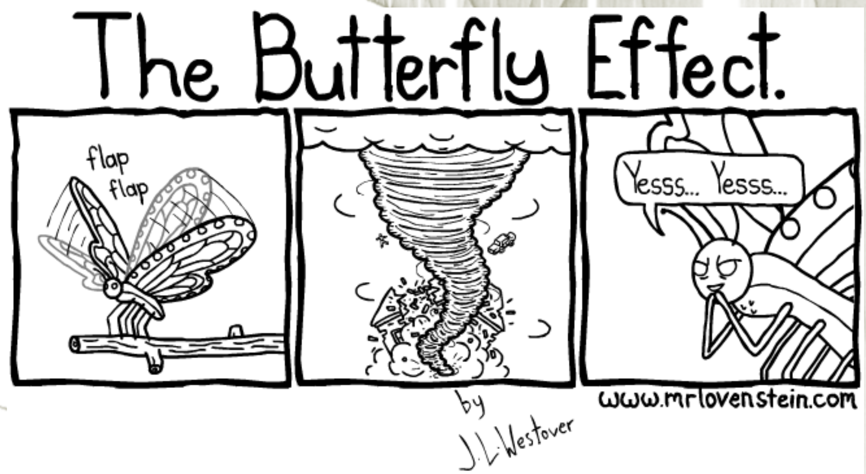
If we knew the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon has been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomenon. A small error in the former well produce an enormous error in the latter. Prediction becomes impossible…

Henri Poincare in *Science and Method* as highlighted by Mitchell.

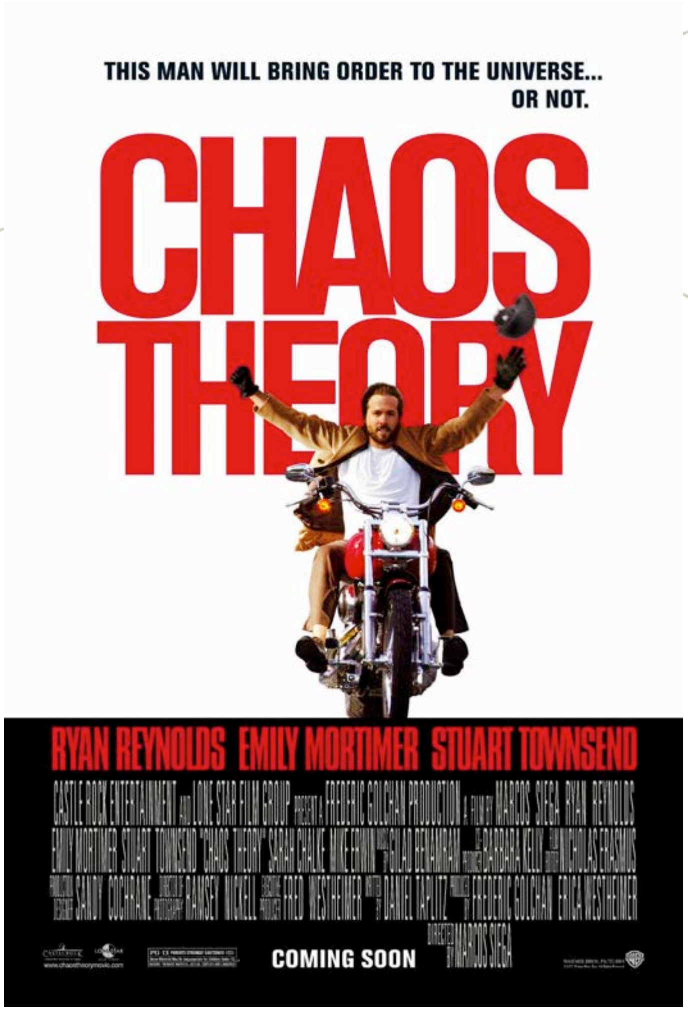
Brief History of Dynamics

Brief History of Dynamics

- This property became known as “sensitive dependence on initial conditions”.
- It became popularized as the “butterfly effect”
- “Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?” Title of a talk given by Lorenz in 1972.

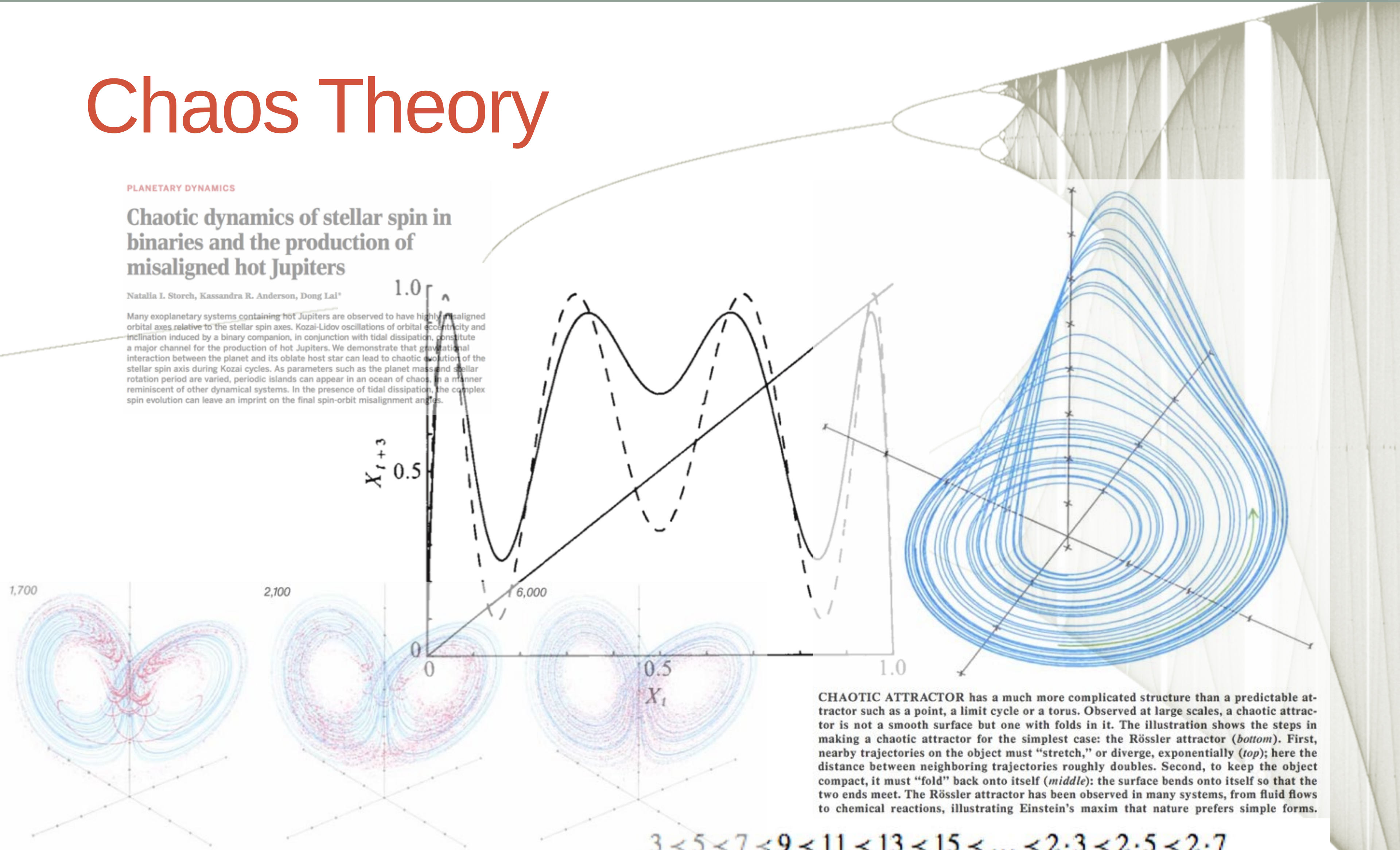


Brief History of Dynamics



- The study of such systems became known as "Chaos Theory"
- Chaos Theory seeks to understand systems that are non-probabilistic but are not predictable.
- It is a branch of dynamics.
- Became popular after computers became accessible to university mathematicians.

Chaos Theory



CHAOTIC ATTRACTOR has a much more complicated structure than a predictable attractor such as a point, a limit cycle or a torus. Observed at large scales, a chaotic attractor is not a smooth surface but one with folds in it. The illustration shows the steps in making a chaotic attractor for the simplest case: the Rössler attractor (*bottom*). First, nearby trajectories on the object must "stretch," or diverge, exponentially (*top*); here the distance between neighboring trajectories roughly doubles. Second, to keep the object compact, it must "fold" back onto itself (*middle*): the surface bends onto itself so that the two ends meet. The Rössler attractor has been observed in many systems, from fluid flows to chemical reactions, illustrating Einstein's maxim that nature prefers simple forms.

$$3 < 5 < 7 < 9 < 11 < 13 < 15 < \ldots < 2\cdot3 < 2\cdot5 < 2\cdot7$$
$$\ldots < 2\cdot2\cdot3 < 2\cdot2\cdot5 < 2\cdot2\cdot7$$
$$\ldots < 2\cdot2\cdot2\cdot3 < \ldots < 2^5 < 2^4 < 2^3 < 2^2 < 2 < 1.$$

DIVERGENCE of nearby trajectories is the underlying reason chaos leads to unpredictability. A perfect measurement would correspond to a point in the state space, but any real measurement is inaccurate, generating a cloud of uncertainty. The true state might be anywhere inside the cloud. As shown here for the Lorenz attractor, the uncertainty of the initial measurement is represented by 10,000 red dots, initially so close together that they are indistinguishable. As each point moves under the action of the equations, the cloud is stretched into a long, thin thread, which then folds over onto itself many times, until the points are spread over the entire attractor. Prediction has now become impossible: the final state can be anywhere on the attractor. For a predictable attractor, in contrast, all the final states remain close together. The numbers above the illustrations are in units of 1/200 second.

Examples of Dynamical System

- Probably you are most familiar with Dynamics in the case of Newton's Laws

$$\sum \vec{F}_{ext} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}$$

- These are *Differential Equations* and are hard to solve.
- Not only are they hard to solve analytically, they are difficult to numerically solve.

Examples of Dynamical Systems

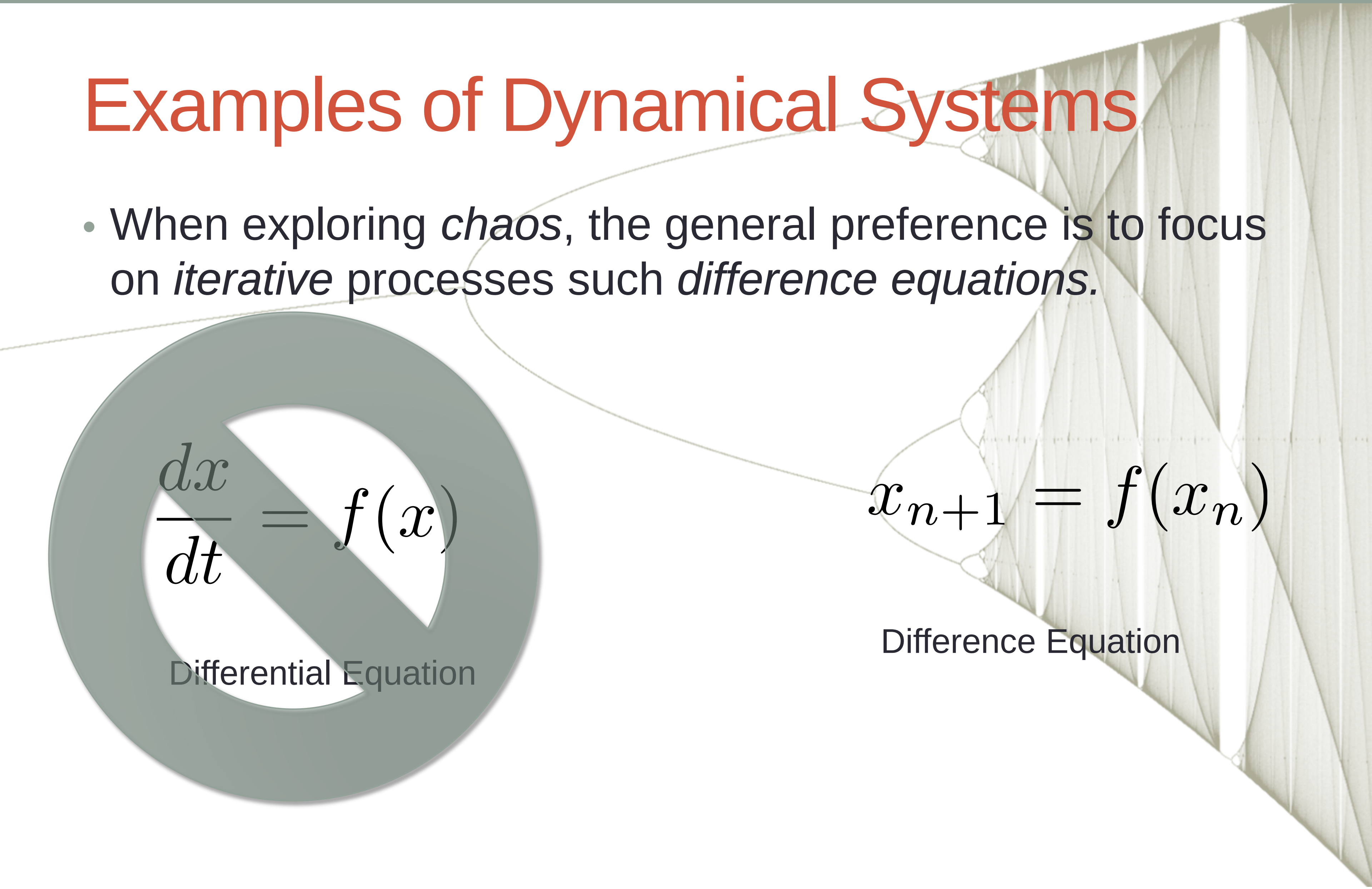
- When exploring *chaos*, the general preference is to focus on *iterative* processes such *difference equations.*

$$\cancel{\frac{dx}{dt} = f(x)}$$

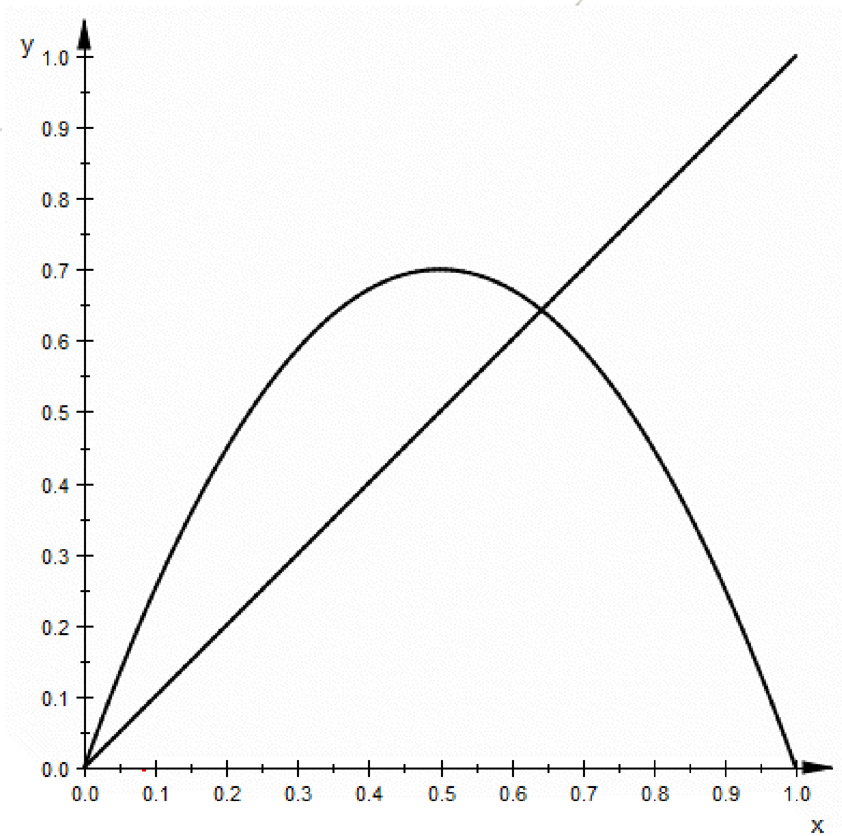
Differential Equation

$$x_{n+1} = f(x_n)$$

Difference Equation

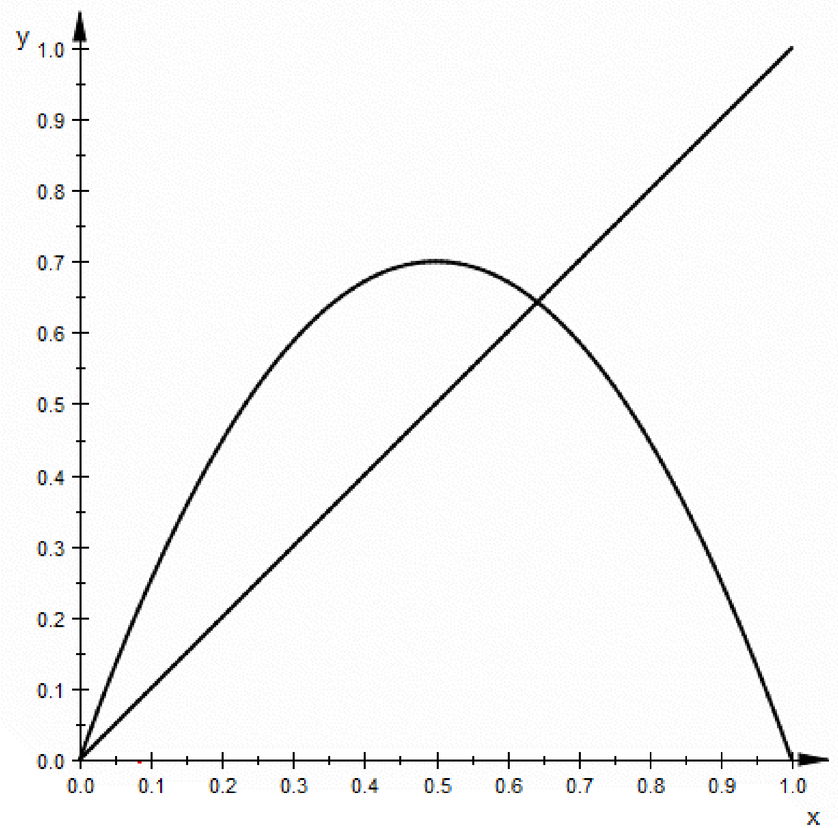


Graphical Analysis



- Sometimes called the *cobweb* diagram
- [Java Applet]()

Graphical Analysis



1. Draw a vertical line from x0 on the x-axis to the function.
2. Without raising your pencil, draw a horizontal line from the function to y=x.
3. Without raising your pencil, draw a vertical line from y=x to the function.
4. Goto Step 2

Simple Economics Example

- *Iterative* processes are much easier to deal with both computationally and algebraically.
- One example comes from finance.
- Imagine you have an account with initial investment of 1M HKD with an annual return rate of 10%.
- How much money will be in your account next year?

$$V_1 = V_0 + 0.10V_0 = 1.10V_0$$

- Or 1.1M HKD…
- How much money will be in your account in the 10th year?

$$V_{10} = V_9 + 0.10V_9 = 1.10V_9$$

Simple Economics Example

- In general, the amount of money in your account in a given year is easy to calculate if you know the amount in the previous year.

$$V_n = V_{n-1} + 0.10V_{n-1} = 1.10V_{n-1}$$

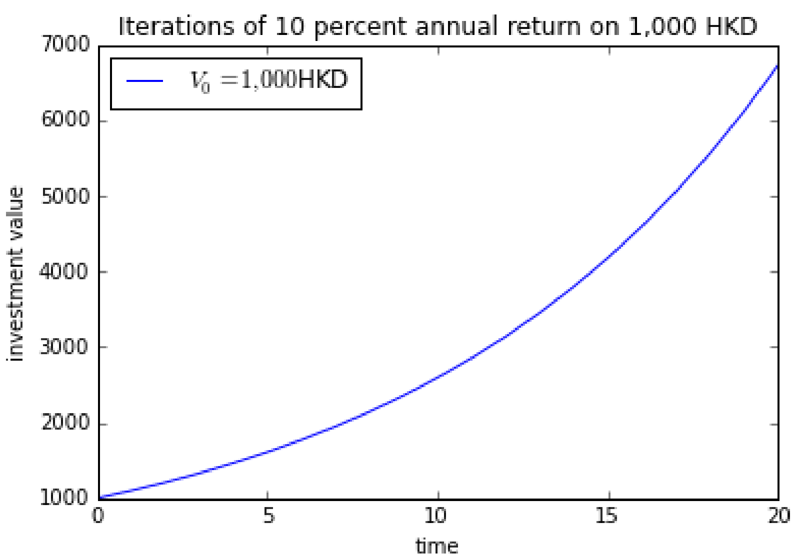
- Of course the only way to know in how much money we have in the n^{th}-1 year is to look at the n^{th}-2 year and so on all the way down to zero.

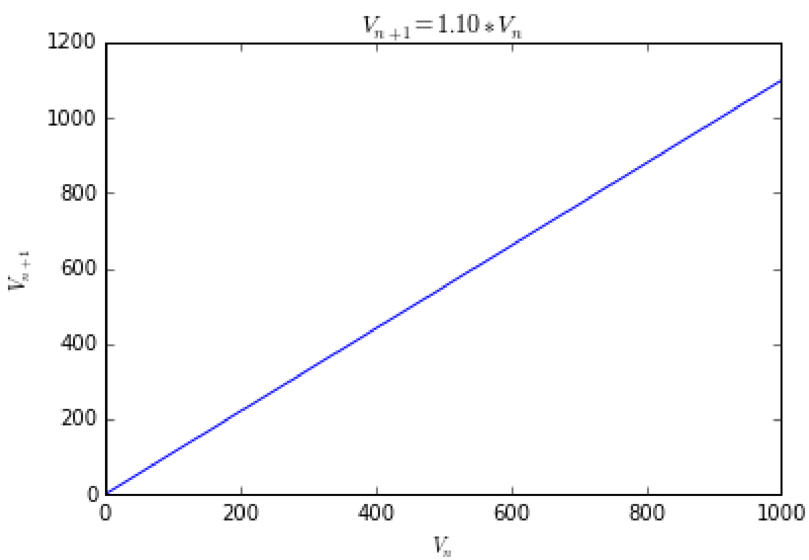
Simple Economics Example

$$
\begin{aligned}
V_n &= 1.10 V_{n-1} \\
&= 1.10 \times 1.10 V_{n-2} = 1.10^2 V_{n-2} \\
&= 1.10^2 \times 1.10 V_{n-3} = 1.10^3 V_{n-3} \\
&\ldots \\
&= 1.10^n V_{n-n} = 1.10^n V_0
\end{aligned}
$$

Simple Economics Example

- Although the growth of the account is *exponential* we will say that the dynamical system is *linear* as the amount of money in the account one year is a *linear* function of the amount in the previous year.



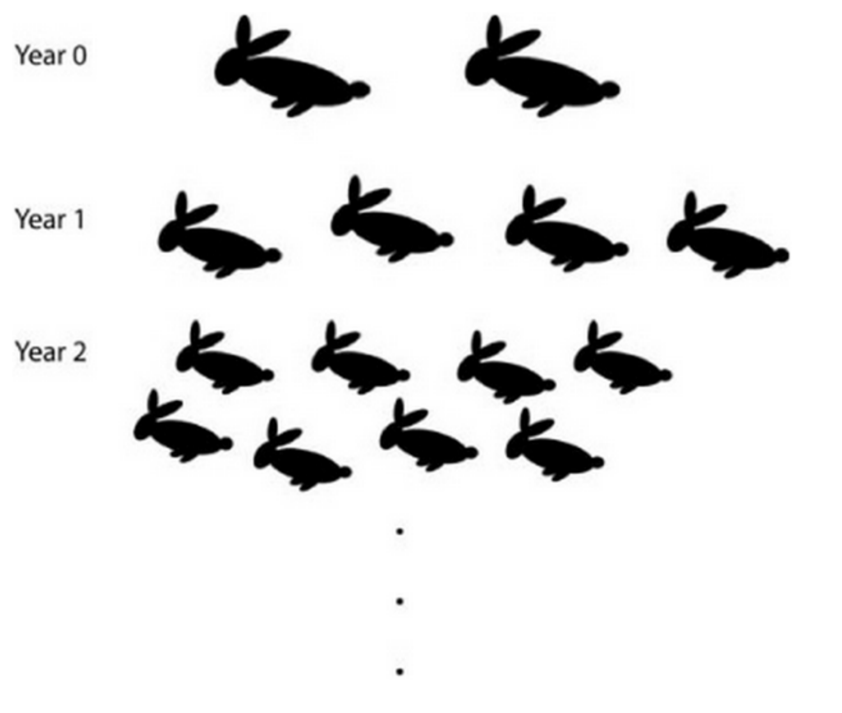


Simple Economics Example

- A consequence of the *linearity* of this system, is that we get the same exact results if we model 1 person with 1M HKD, 1000 people with 1000HKD each or 1000000 people with 1 HKD each.
- In any case, the whole is equal to the sum of the parts.

Population Dynamics

- A similar model has been used in the past to describe Population Dynamics
- Simply switch HKD to bunnies and Annual Return to Birth Rate.
- Unrealistic as there are only two outcomes… The exponential decay of the species or the exponential growth.



Population Dynamics

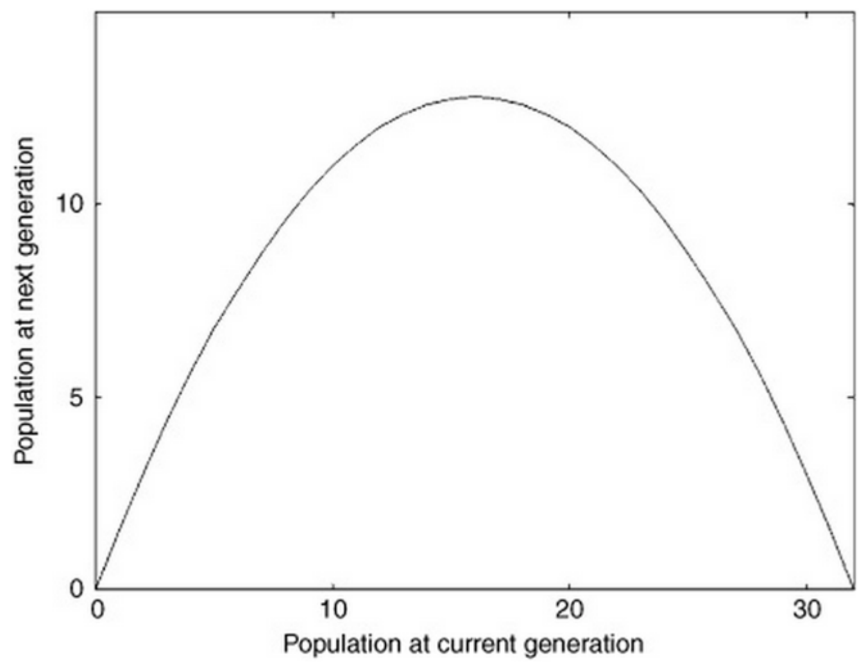
- The *Logistic Model* of population growth is an attempt to make a more realistic model.
- It includes 3 parameters
  - Birthrate (BR) = ratio of offspring to parents
  - Deathrate (DR) = ratio of offspring who don't survive to parents
  - Carrying Capacity (CC) = The maximum population supported by the habitat

$$P_{t+1} = (BR - DR)\frac{P_t(CC - P_t)}{CC}$$

Population Dynamics

$$P_{t+1} = (BR - DR)\frac{P_t(CC - P_t)}{CC}$$

$$\begin{aligned}
P_0 &= 20 \\
P_1 &= (2 - 0.4)\frac{20(32-20)}{32} = 12 \\
P_2 &= (2 - 0.4)\frac{12(32-12)}{32} = 12 \\
&\dots
\end{aligned}$$

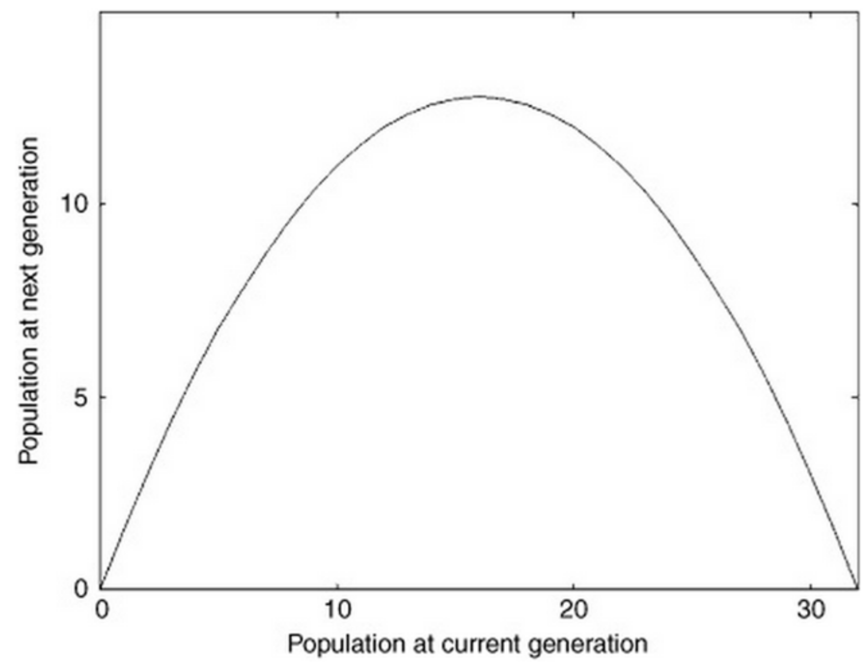


The Logistic Model for BR=2, DR=0.4 and CC=32. For any values, the model will always be a parabola. Taken from *Mitchell.*

Population Dynamics

$$P_{t+1} = (BR - DR)\frac{P_t(CC - P_t)}{CC}$$

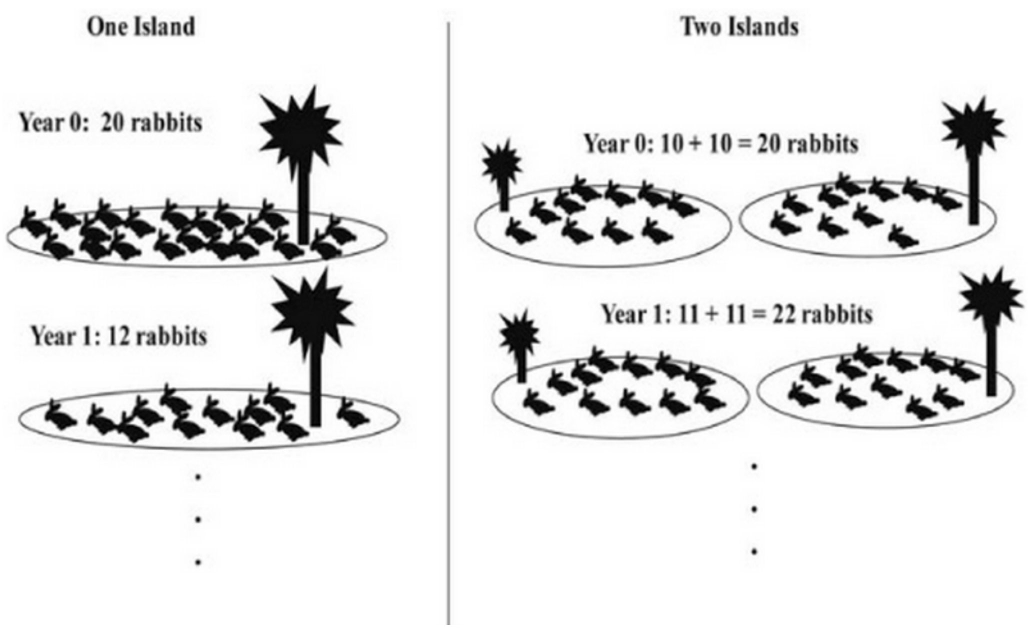
$$\begin{aligned}
P_0 &= 10\\
P_1 &= (2-0.4)\frac{10(32-10)}{32} = 11\\
P_2 &= (2-0.4)\frac{11(32-11)}{32} = 11.55\\
&\dots\\
P_\infty &= 12
\end{aligned}$$



The Logistic Model for BR=2, DR=0.4 and CC=32. For any values, the model will always be a parabola. Taken from *Mitchell.*

Population Dynamics

$$P_{t+1} = (BR - DR)\frac{P_t(CC - P_t)}{CC}$$



In the previous linear model, it didn't matter how we split the initial population. Here we end up with a very different end result if we split the population amongst other parts. The whole is no longer the sum of its parts.

The Logistic Map

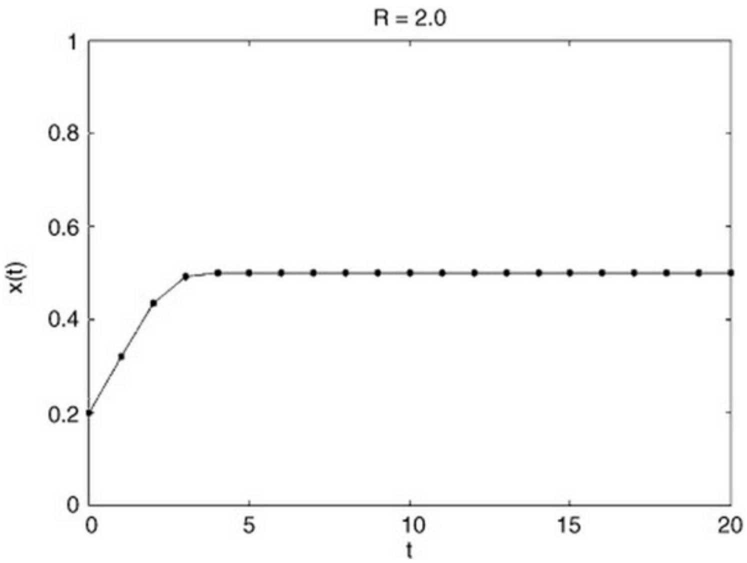
$$P_{t+1} = (BR - DR)\frac{P_t(CC - P_t)}{CC}$$

- Eventually the mathematicians would get their hands on this.
- Combine (BR-DR)/CC into one parameter R.
- Replace P with P/CC and call it x

$$x_{n+1} = R x_n (1 - x_n)$$

Gradually any relationship with populations became a fable and people were interested in the logistic map for its properties as a pure mathematical object.

The Logistic Map



This graph shows the orbit of x_0=0.2 when R=2.0.

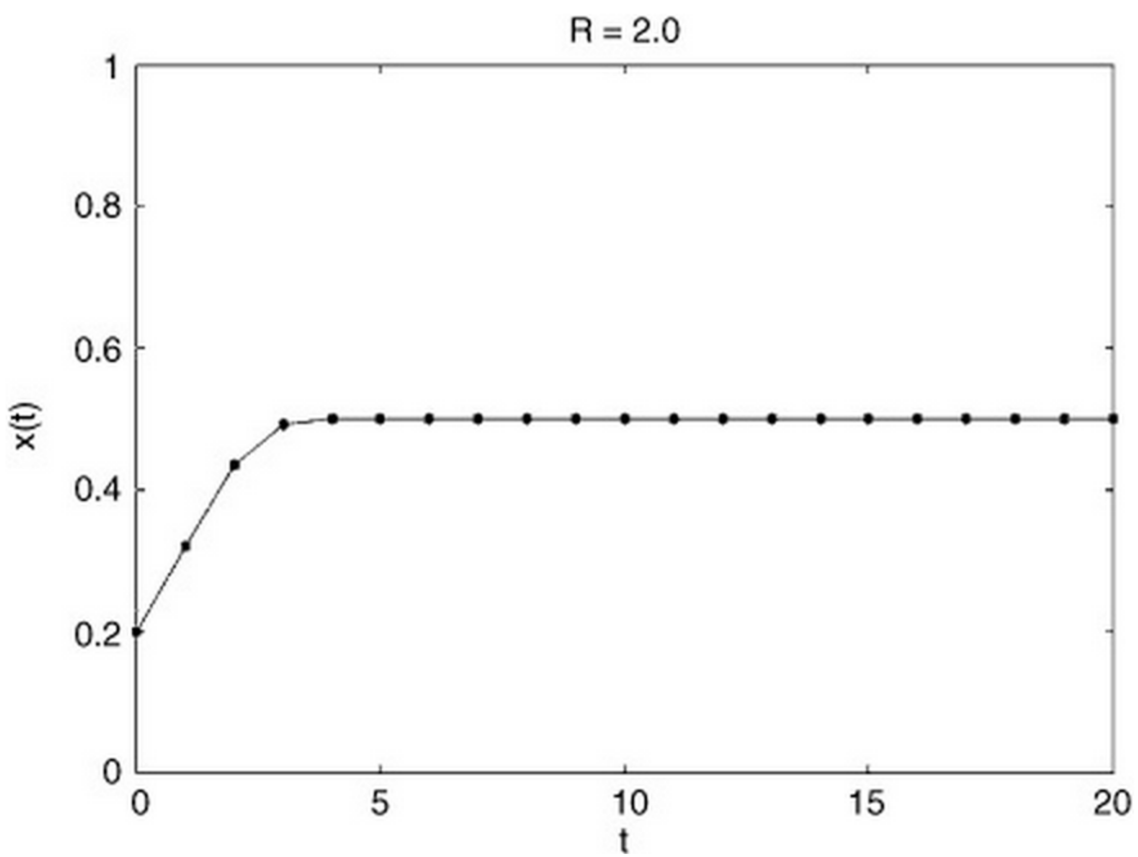
- We generally have one question.
- For a given value of R, how do different values of x_0 behave when we iterate the model.
- How a value of x_0 behaves when we iterate the model is often called its *orbit* or *trajectory.*

The Logistic Map

We generally have one question.

For a given value of R, how do different values of x0 behave when we iterate the model.

How a value of x0 behaves when we iterate the model is often called its *orbit* or its *trajectory*.

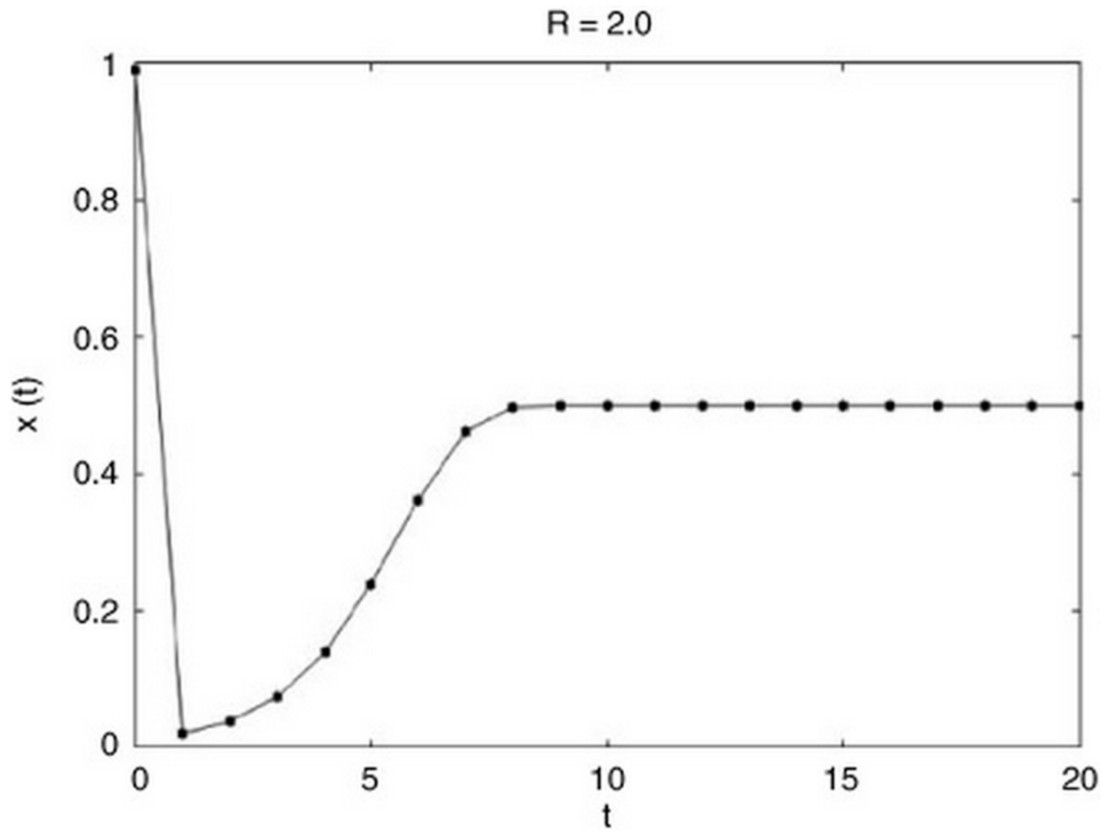


The Logistic Map

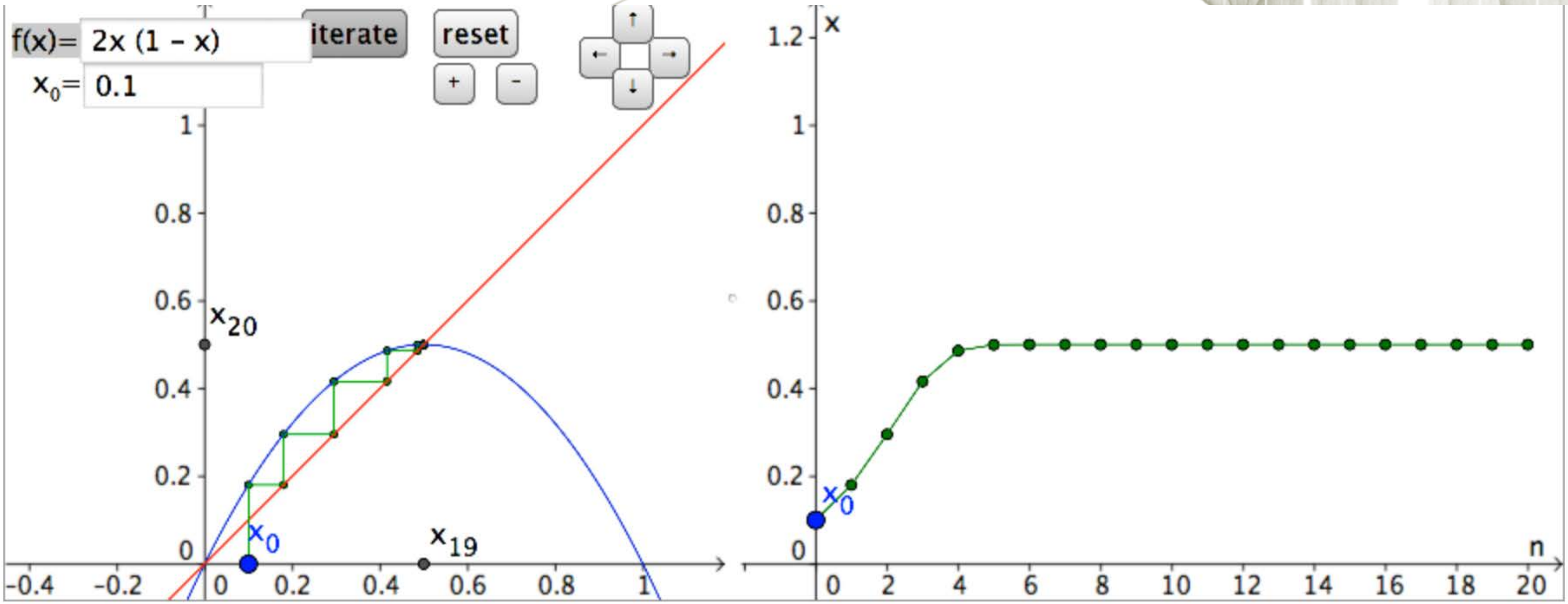
Here we pick x_0=0.99 while keeping R=2.0. We see that the trajectory goes to 0.5 again.

For R=2.0, x=0.5 is a *fixed point*.

Since points “nearby” to 0.5 have orbits that approach 0.5, we say that 0.5 is the *attractor*



The Logistic Map

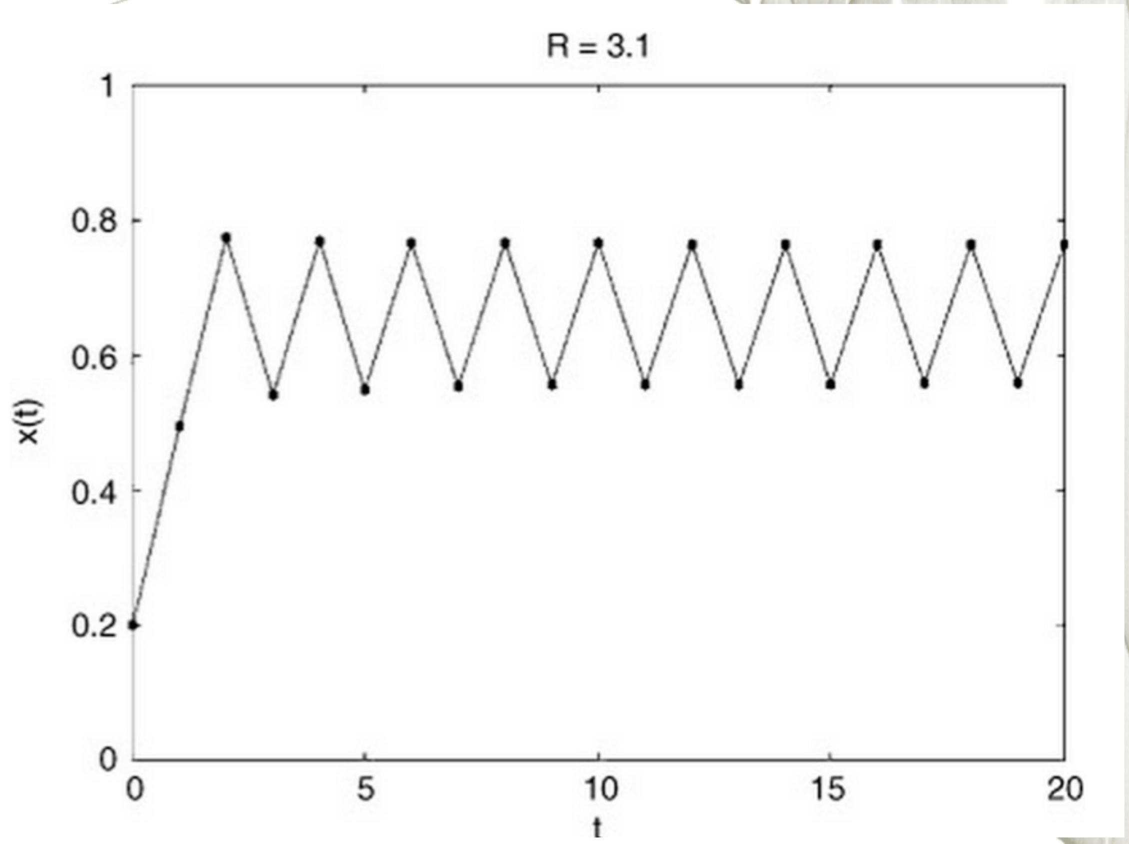


The Logistic Map

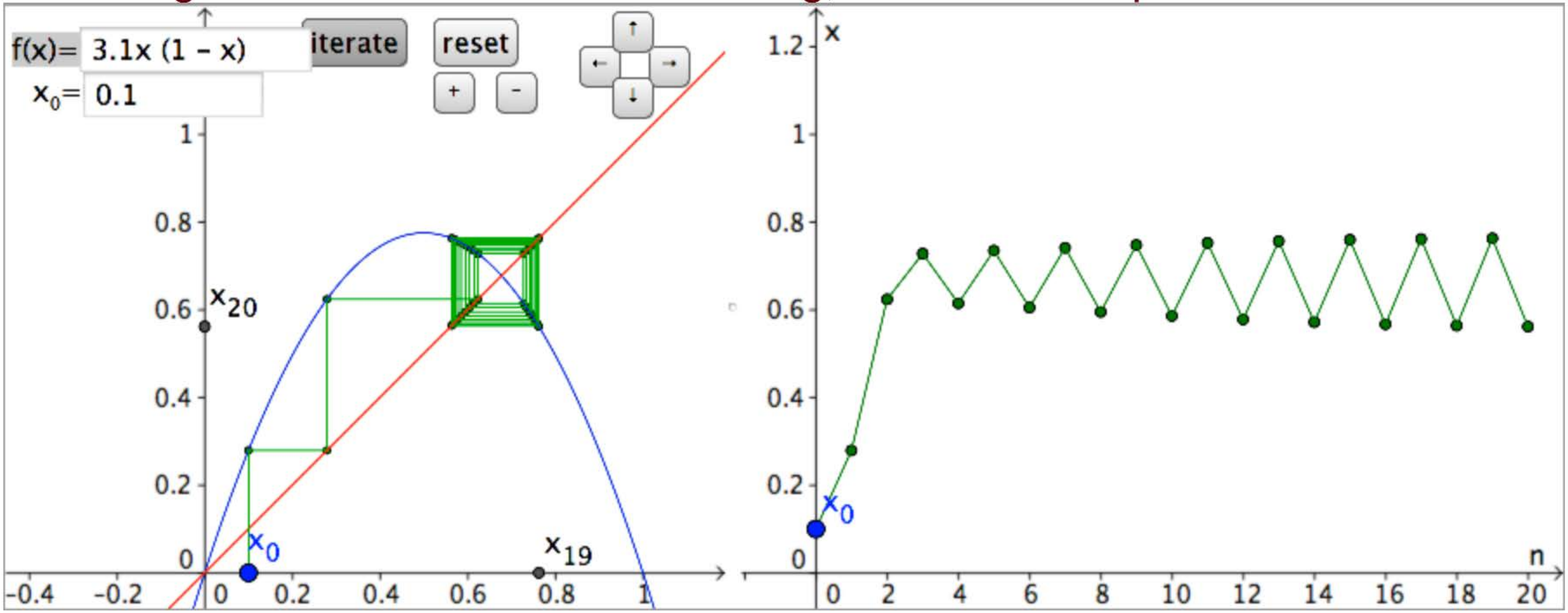
At a different R value, R=3.1 we find a *2-cycle*.

No matter where we start, eventually the orbits of different x_0 values will oscillate near these two points.

This *attracting* 2-cycle is called the *attractor*.



The Logistic Map

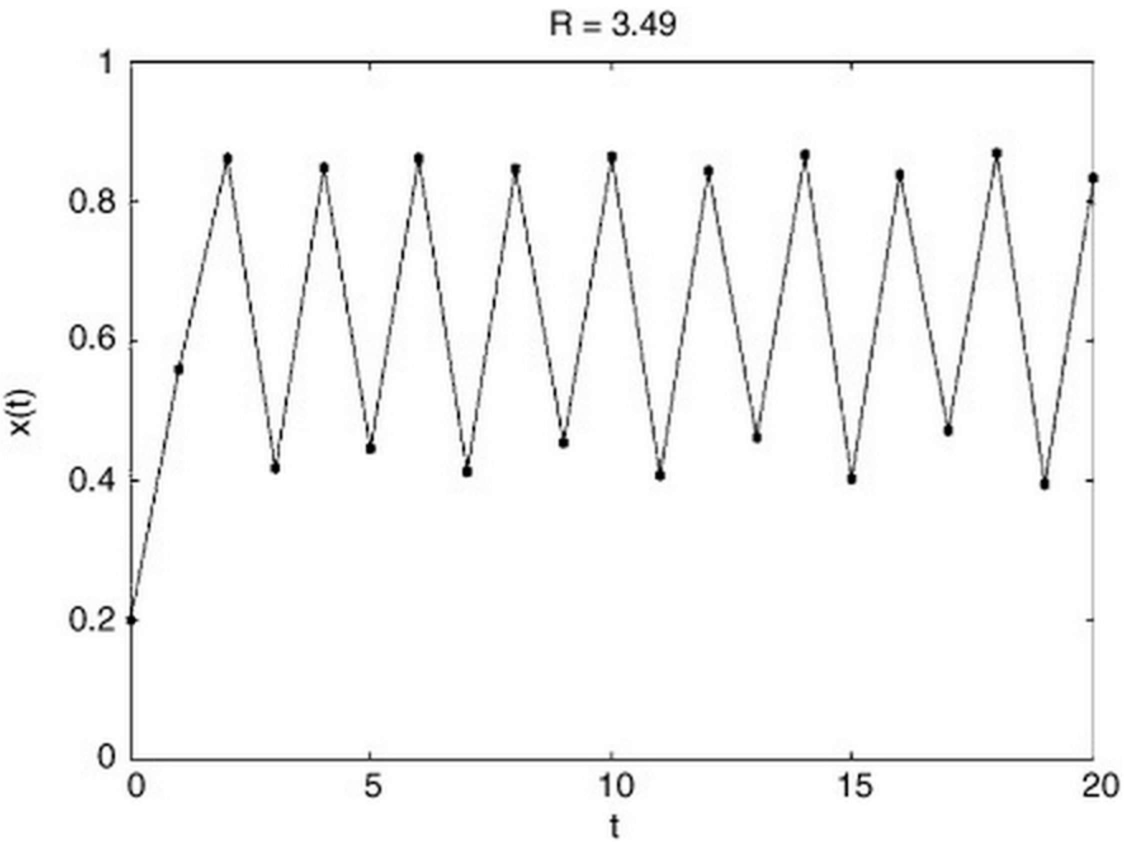


The Logistic Map

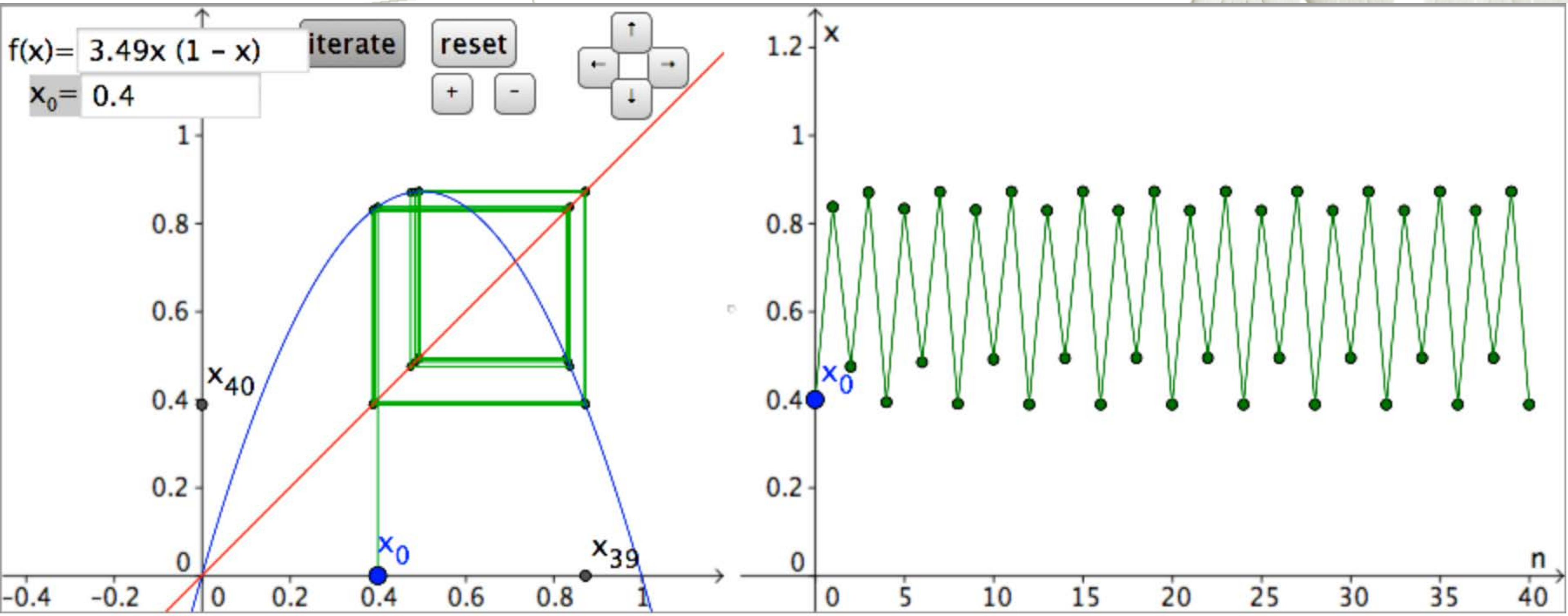
When R=3.49, the *attractor* is a 4-cycle.

We find ourselves on the *period-doubling route to chaos*.

As R gets bigger, the period of the attractor repeatedly doubles and does so faster and faster so that we reach infinite period for a finite R.



The Logistic Map

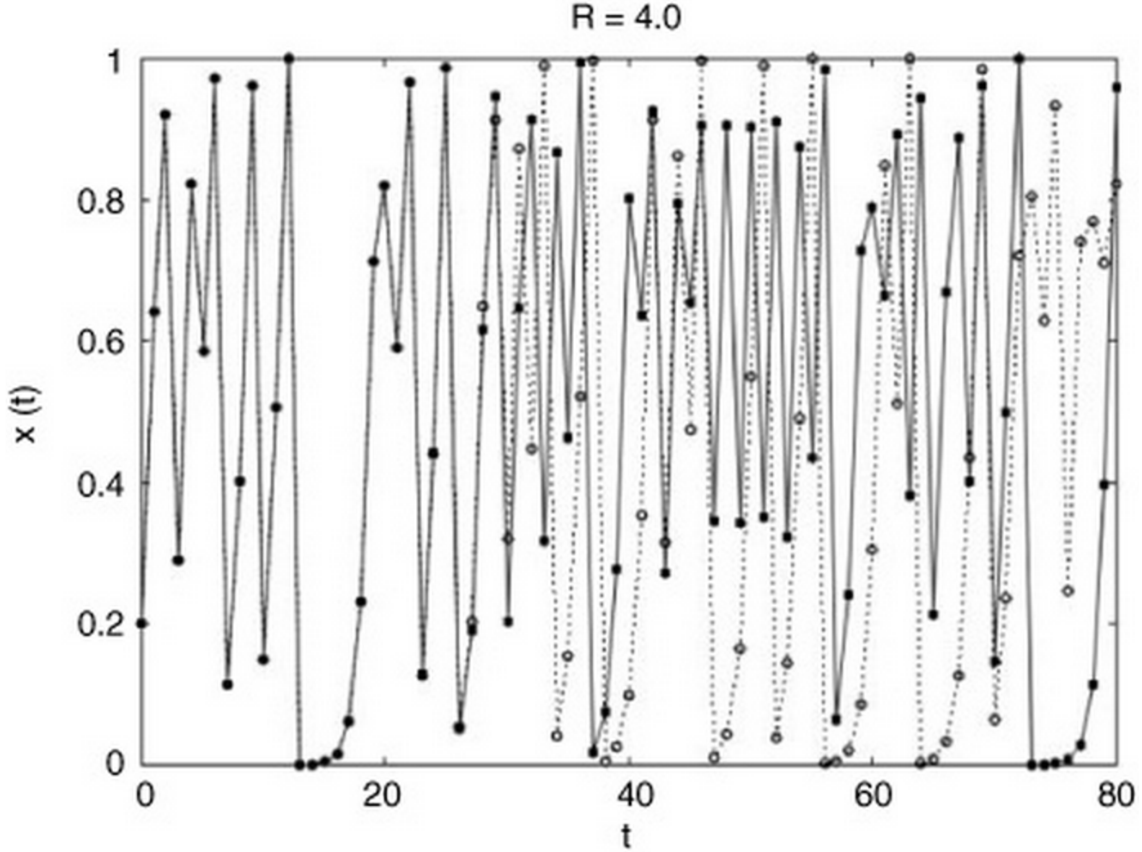


The Logistic Map

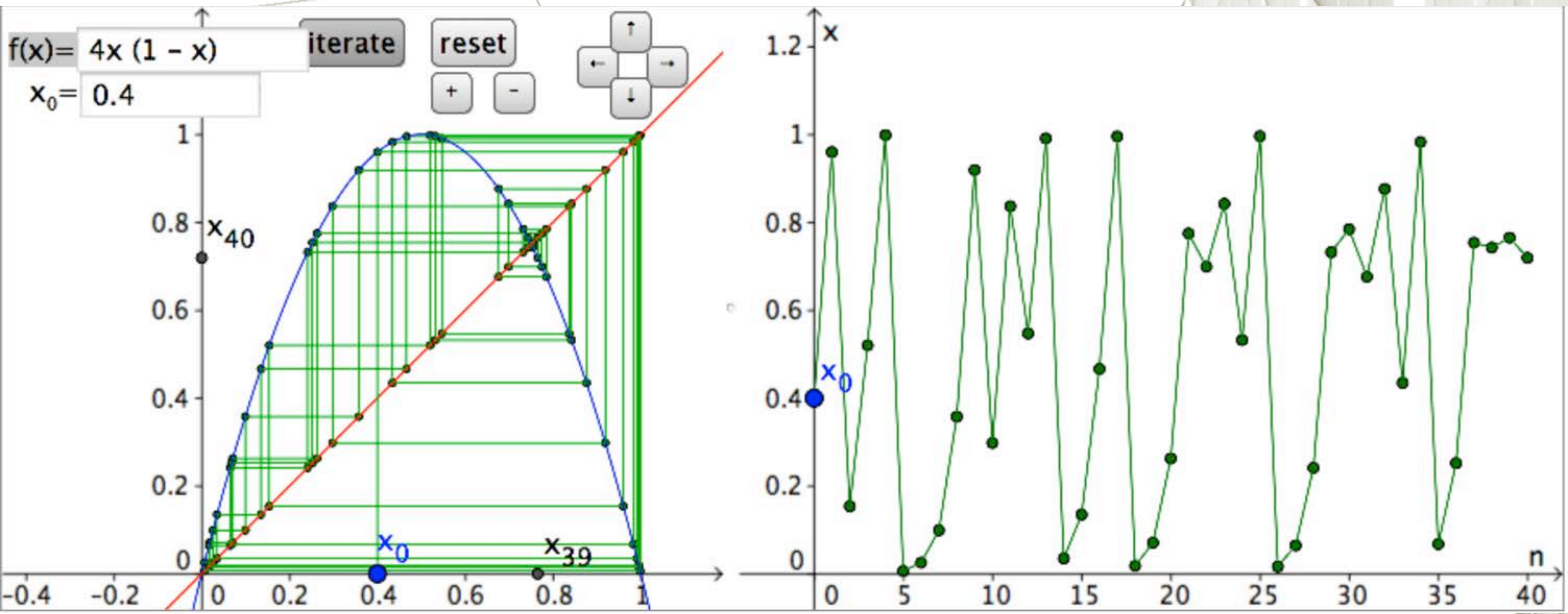
Not only do the dynamics of x become irregular, *sensitive dependence on initial conditions* also emerges.

This graph shows the trajectories of two initial conditions:
x_0=0.2 (solid)
X_0=0.2000000001 (dashed).

At R=4.0, after about 30 iterations the two trajectories are very different.

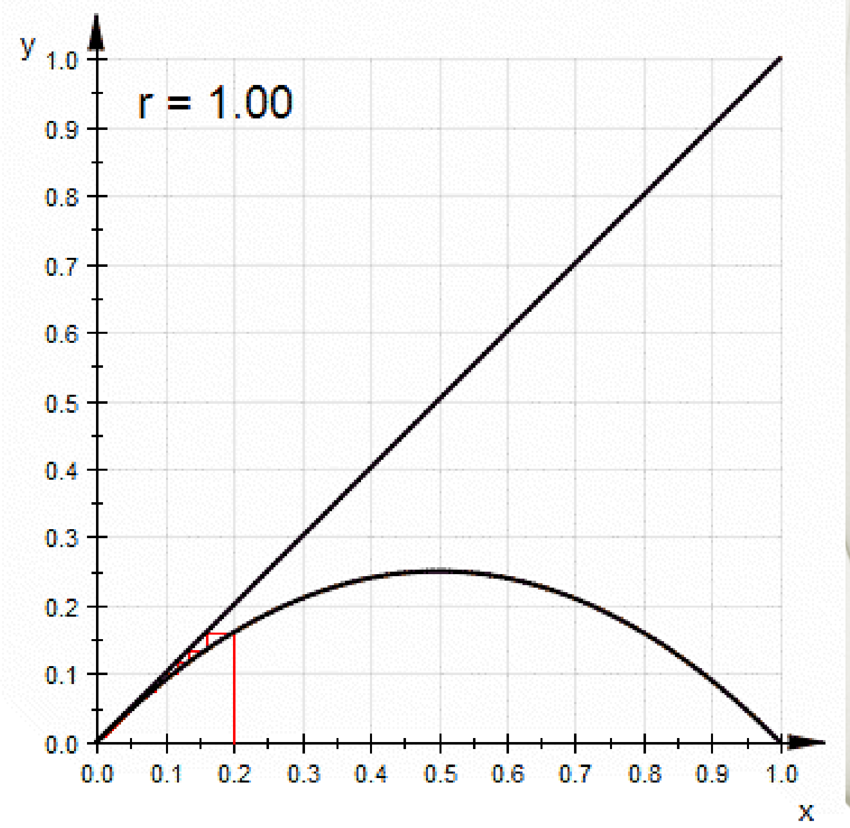


The Logistic Map

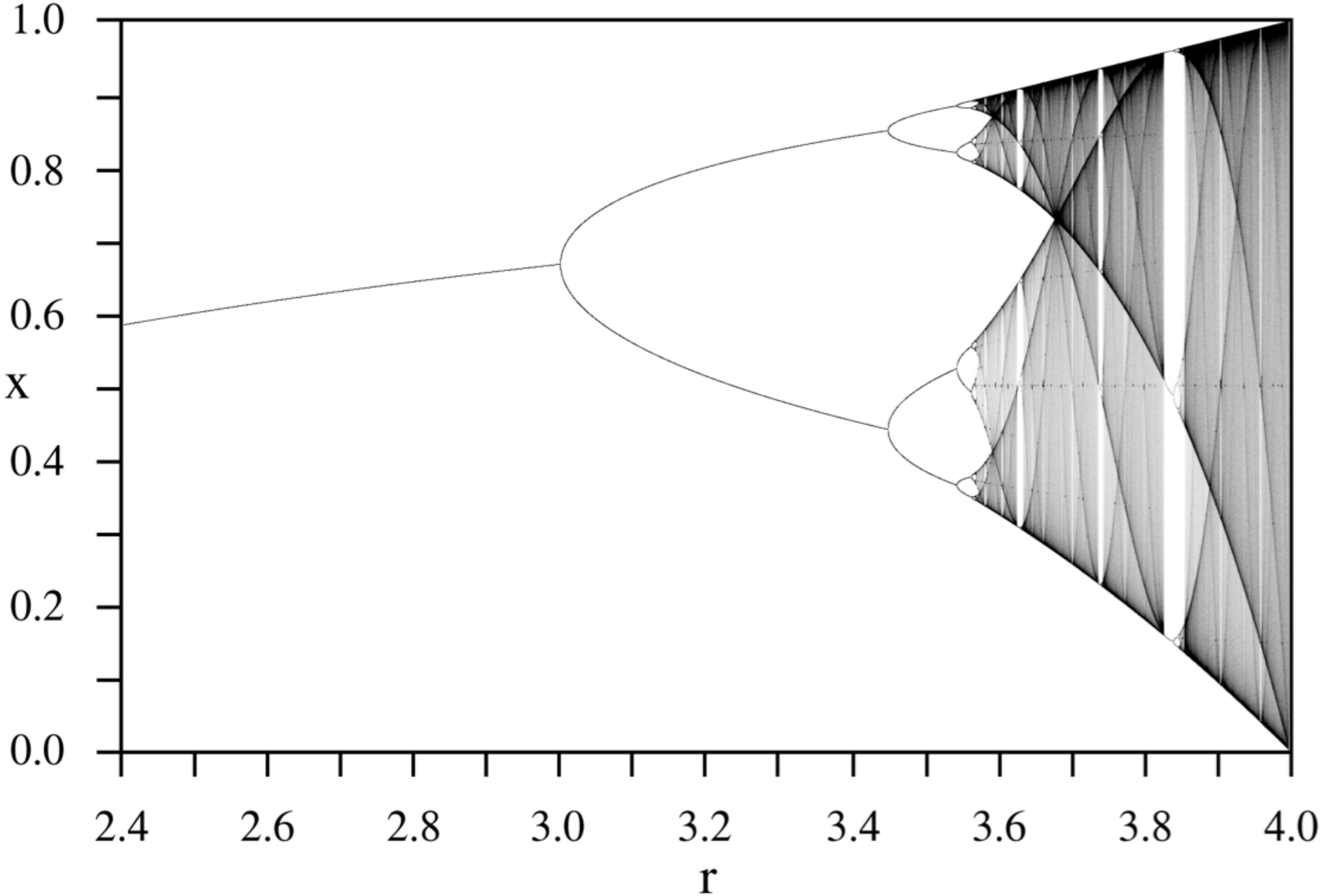


Bifurcation Diagram

- The Bifurcation Diagram gives us a single picture that shows the attractor at different values of the parameter.
- This .gif is nice but is not too suitable for journal publication.



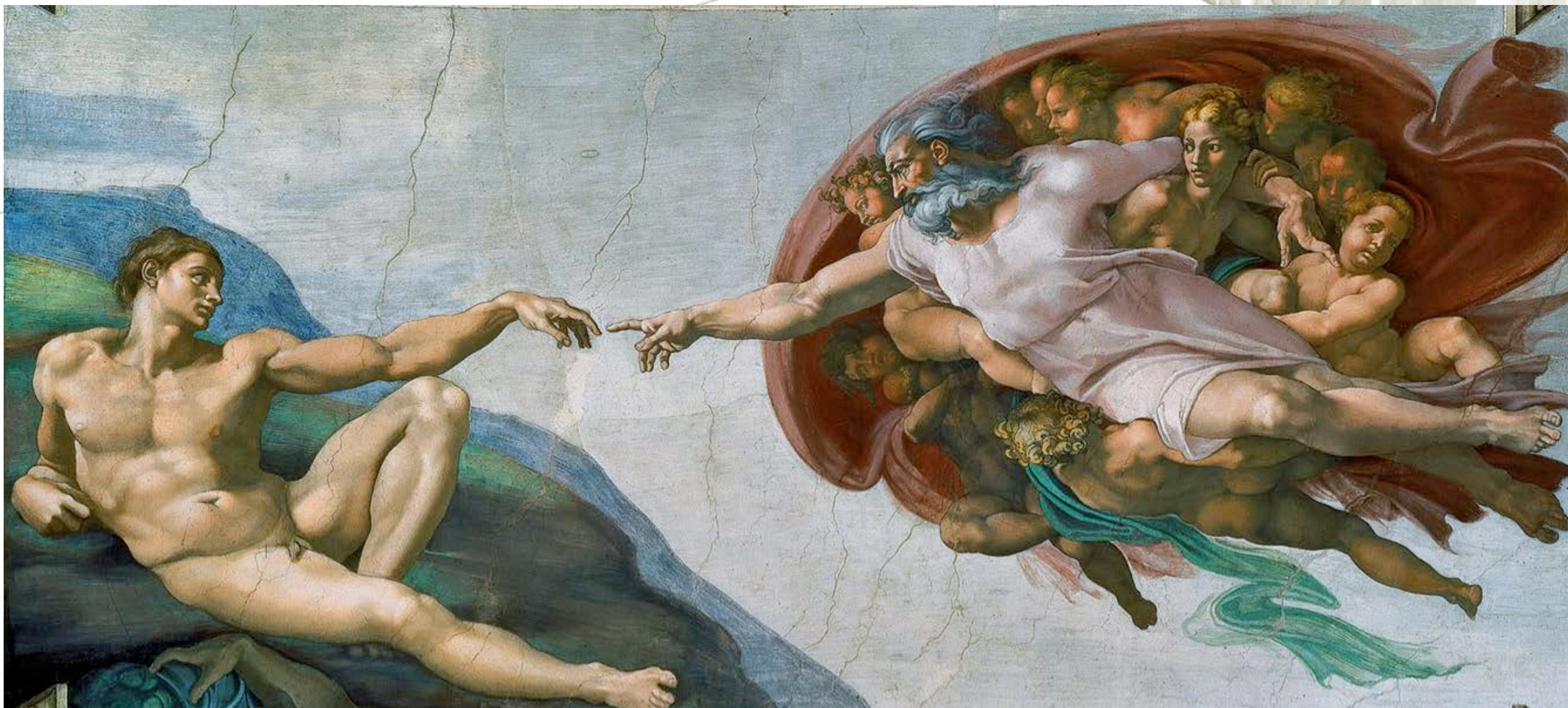
Bifurcation Diagram



Bifurcation Diagram

1. Pick a minimum and maximum value for R (in the case above 2.4 and 4.0 respectively).
2. Set R= to the minimum value.
3. Pick a value for x_0 (since we are interested in the *attractor* this choice is not too important).
4. Iterate the map several times (maybe 100, we call this the transient and gives the system some time to near the attractor).
5. Iterate the map many more times (maybe 1000 times) and record the values in the trajectory.
6. Plot all of the values in the trajectory with a single R value.
7. Slightly increment the value of R (perhaps to 2.4000001).
8. If R is less than the maximum value, go back to step 3, else END.

What's In a Name?



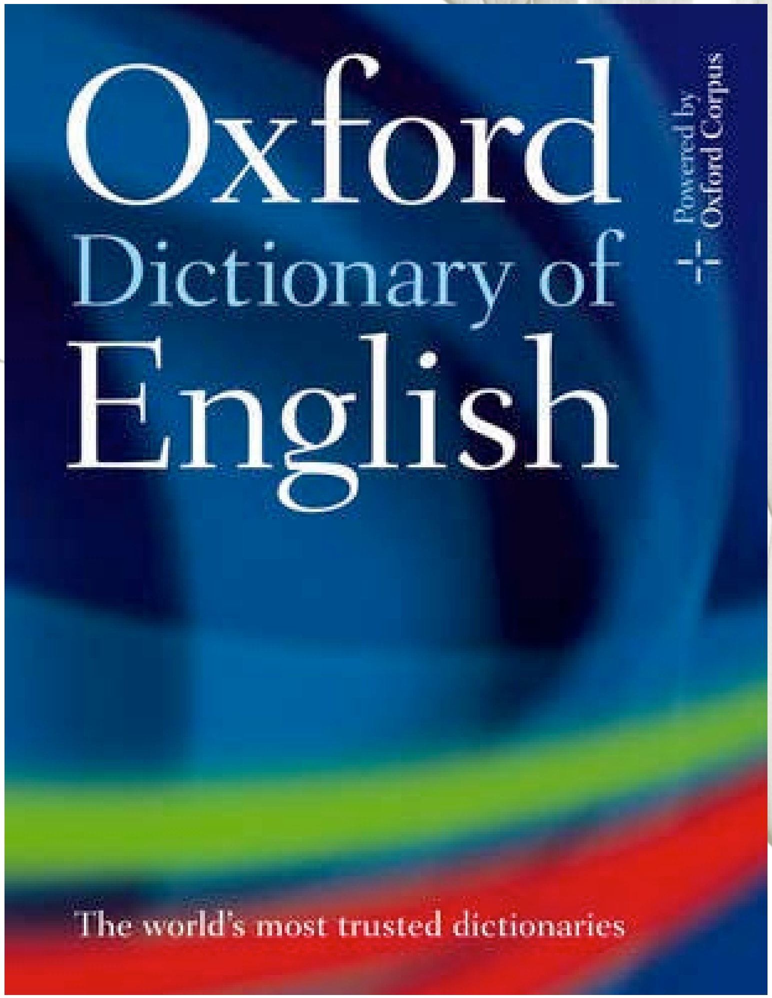
"And the earth was waste and void; and darkness was upon the face of the deep: and the Spirit of God moved upon the face of the waters." Genesis 1:2 American Standard Version. The Hebrew word, "tohu wa-bohu" here traslated "waste and void" is also sometimes translated "chaos and desolation"

What’s In a Name?

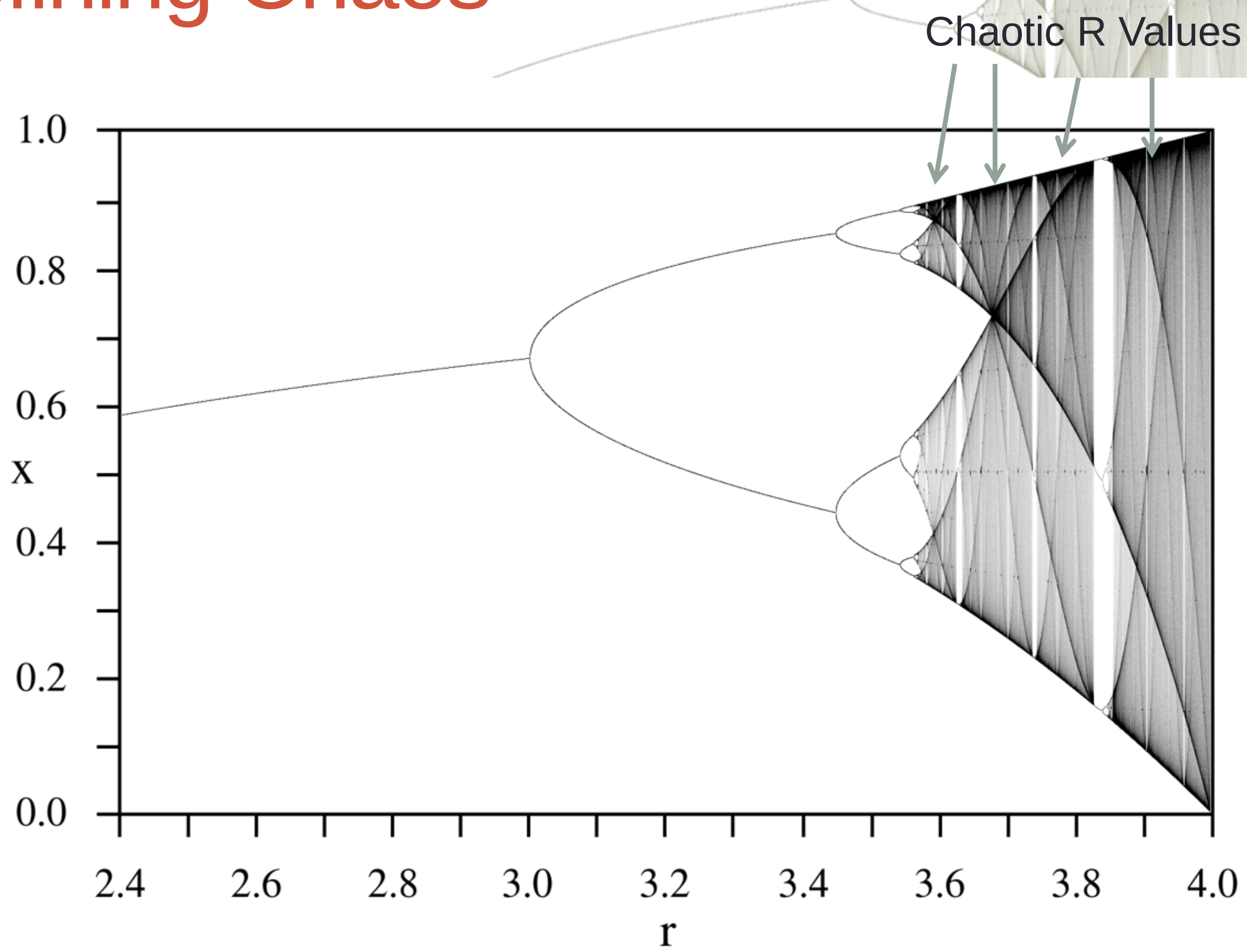
Between 750 and 650 B.C.E., the Greek philosopher Hesiod used the word to refer to the primordial formless state of the universe and personified Chaos as the origin of all other Greek Gods and Titans

What's in a Name

- A gaping void, yawning gulf, chasm, or abyss
- A state resembling that of primitive chaos; utter confusion and disorder.
- *Math.* Behaviour of a system which is governed by deterministic laws but is so unpredictable as to appear random, owing to its extreme sensitivity to changes in parameters or its dependence on a large number of independent variables; a state characterized by such behaviour.



Defining Chaos



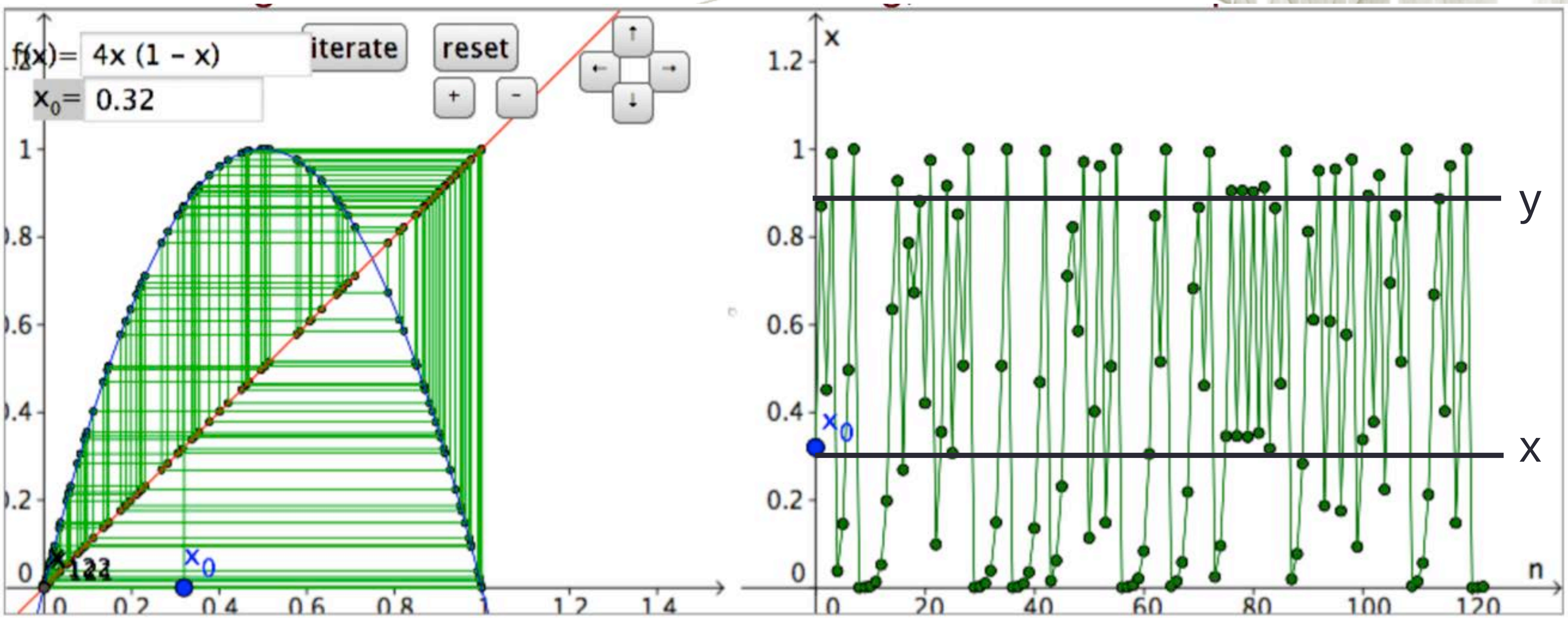
Defining Chaos

- Chaos has a *mathematical* definition in terms of common properties of different dynamical systems that share properties such as *sensitive dependence on initial conditions.*
- The term 'chaos' dates back to 1975 with T. Y. Li and James York's paper, *Period Three Implies Chaos*.
- The most widely used mathematical definition first appeared in the textbook, *An Introduction to Chaotic Dynamical Systems* by Robert Devaney

Defining Chaos

- A function $f{:}X \to X$ is *chaotic* if and only if f satisfies the following three conditions.
  - f is *transitive*
  - The periodic points of f are *dense* on X.
  - f has *sensitive dependence on initial conditions.*

Defining Chaos -- Transitive



The orbit of this x0=0.32 begins very close to x and later passes nearby to the number y. Transitivity says that no matter which x and y I choose or how strictly I define “nearby” I can find such an x0.

Defining Chaos – Transitive

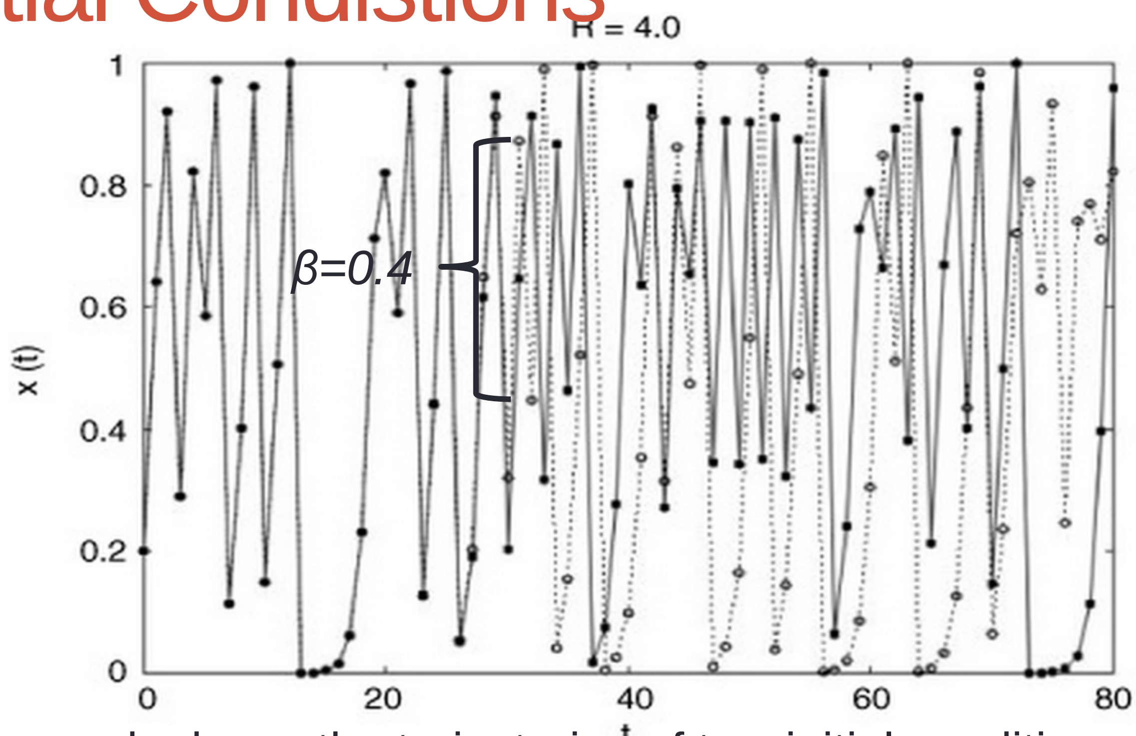
- Systems with this property cannot be split up or decomposed into isolated non-interacting subsets.

- Recall the example from *Jurassic Park*. This requirement would not be satisfied for Ian Malcom's example.

Defining Chaos – Dense Periodic Points

- This requirement gives the existence of regularity in the midst of the complicated.
- It demands that a system that is thought of as chaotic is also quite capable of creating both organized patterns and unpredictable complicated patterns
- Strike Two for Jurassic Park

Defining Chaos – Sensitive Dependence on Initial Condistions

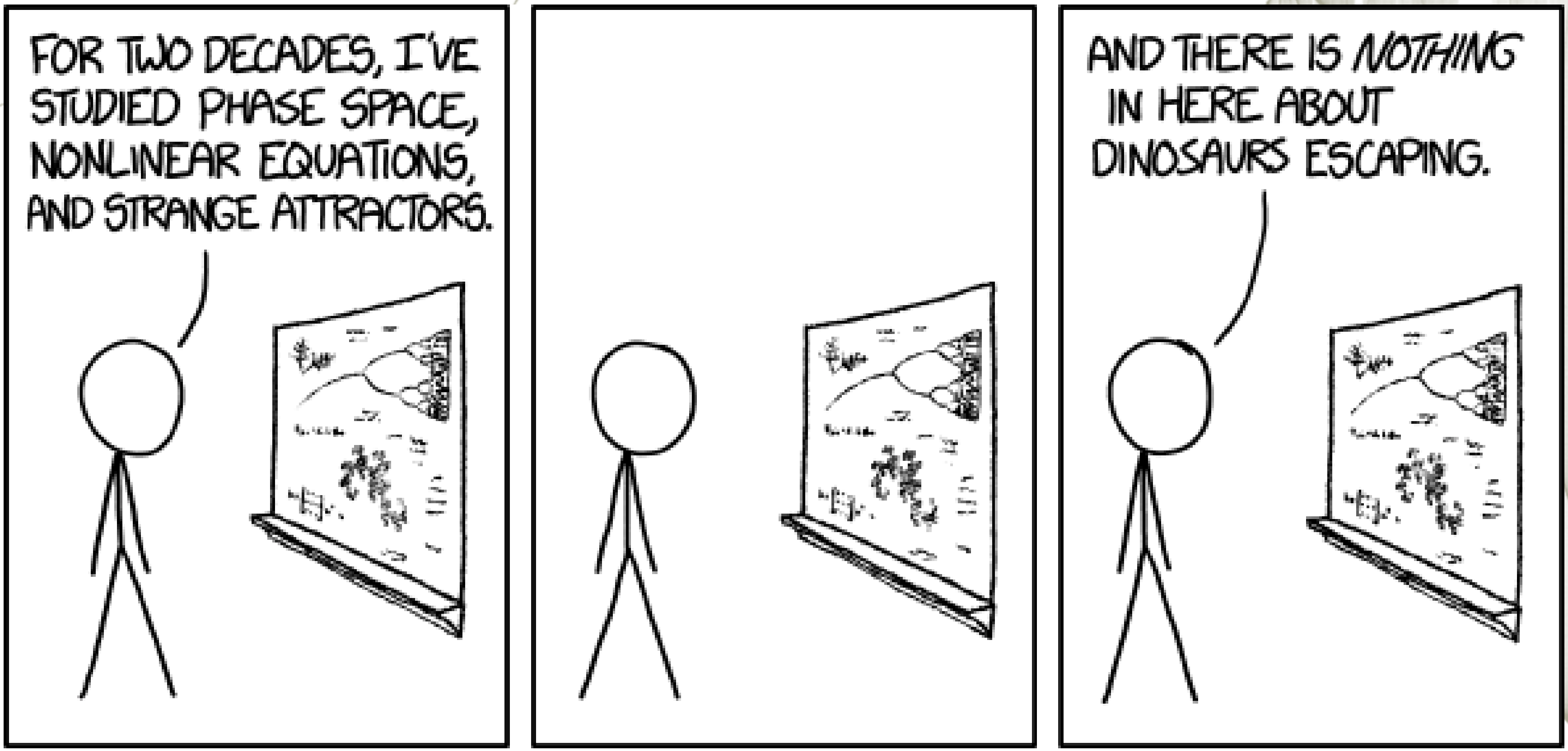


This graph shows the trajectories of two initial conditions:
x_0=0.2 (solid), x_0=0.2000000001 (dashed).
Around the 35th iteration, their orbits differ by more than 0.4.
Even if we picked much closer x0 values and a bigger *β,* we could find some iteration k where they were separated that much.

Defining Chaos – Sensitive Dependence on Initial Conditions

- This requirement shows that these systems exhibit unpredictability. It is our most intuitive requirement.
- I'll grant Jurassic Park this case.

Defining Chaos



Lessons from Chaos

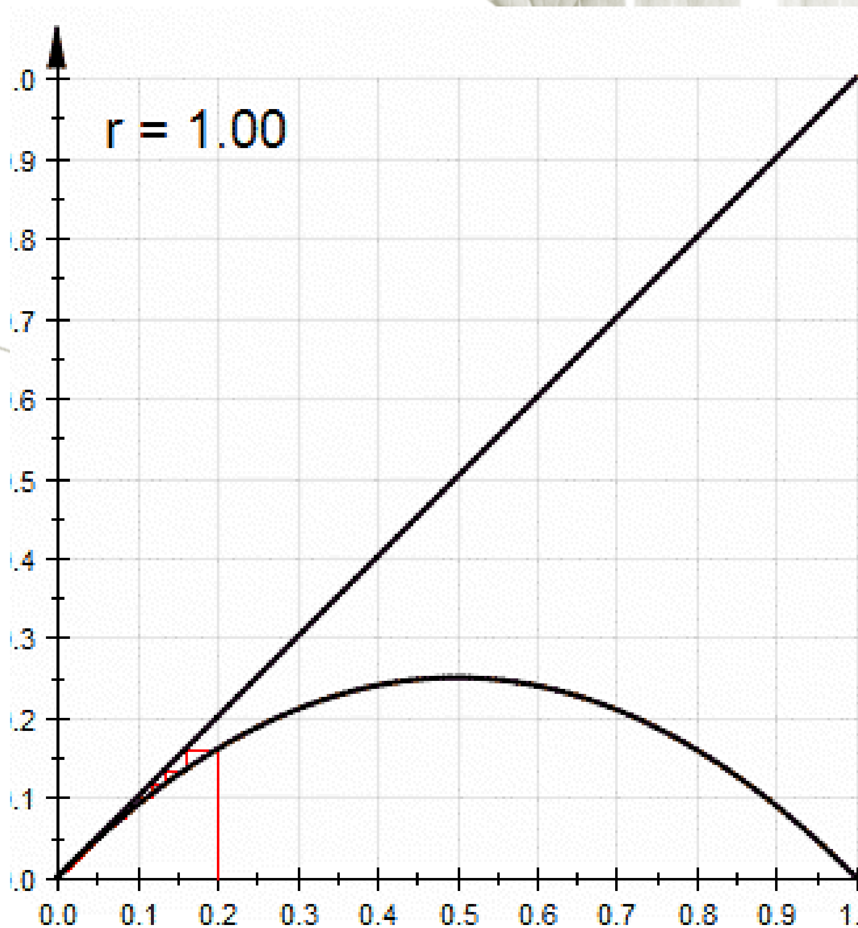
- Seemingly random behavior (infinite cycles) can occur from *deterministic* rules.
- There need be no external source of chance.

Lessons from Chaos

- The long term behavior of systems can be impossible to predict in principle… not just in practicality.

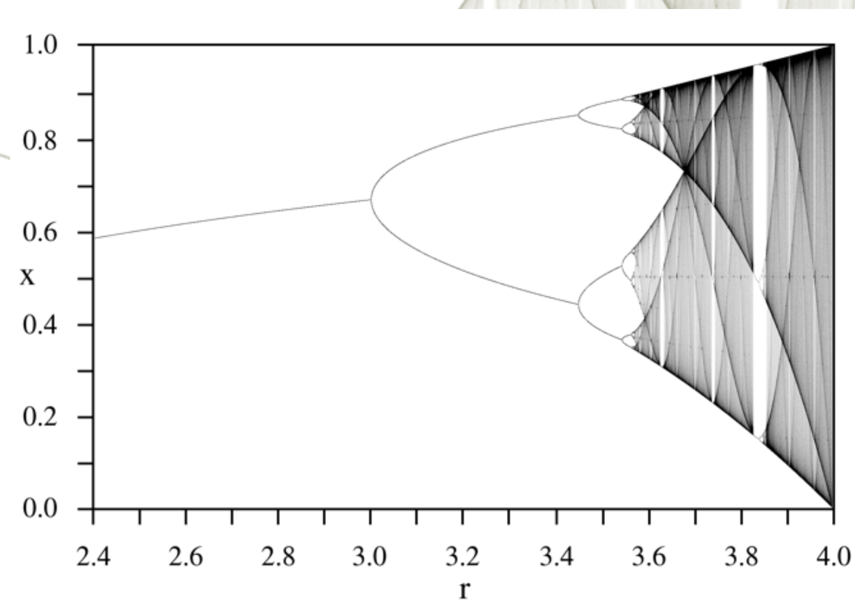
Lessons from Chaos

- A single natural rule can create both regular patterns and complicated behaviors



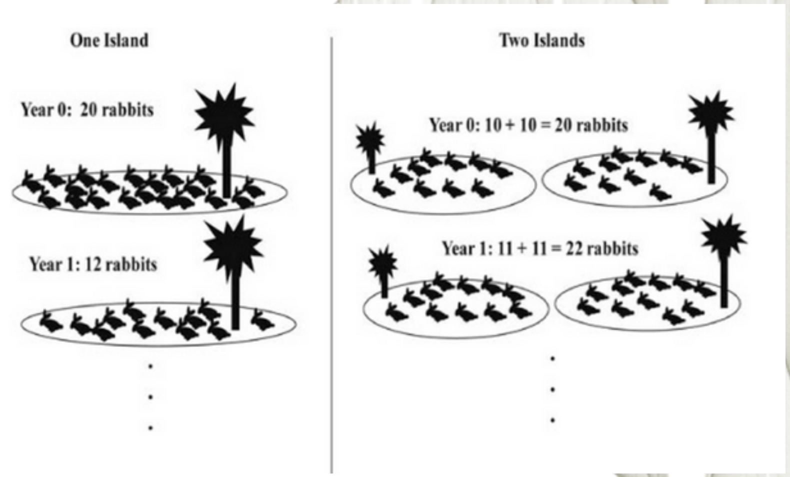
Lessons from Chaos

- Some properties of such systems are universal.
- There are some aspects of such systems which are inherently predictable.



Chaos and Complexity – The Logistic Model

- Ignores specific reproductive habits, predator/prey relationships, local environmental affects
- Ignores the workings of the members of the populations itself
- Is a “top-down” approach
- Exhibits complex collective behavior and emergent properties



Looking Ahead

- Models that are very simplified may be good enough to give us the important ideas of natural systems.
- This will form the basis of the future of this course.